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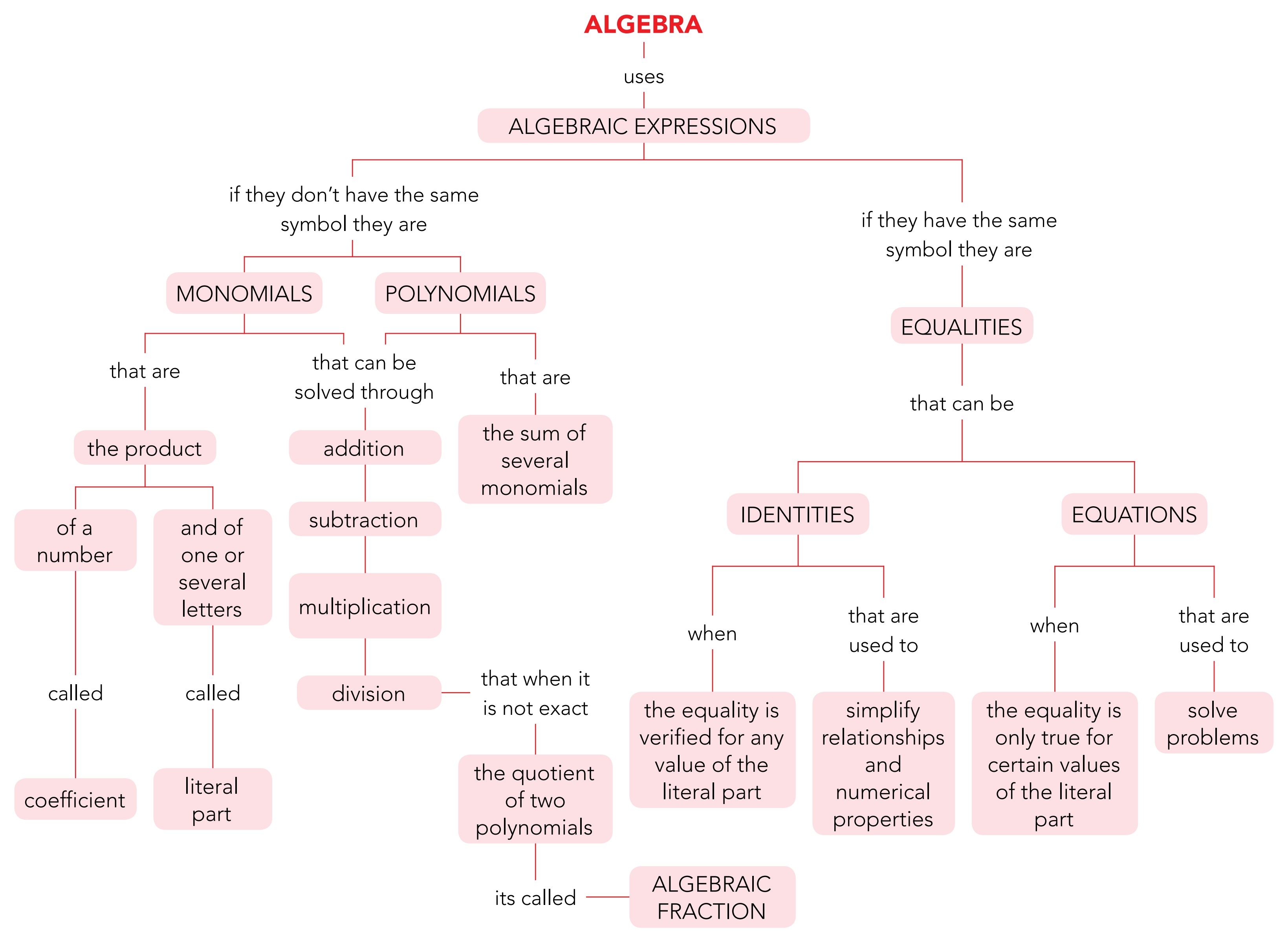
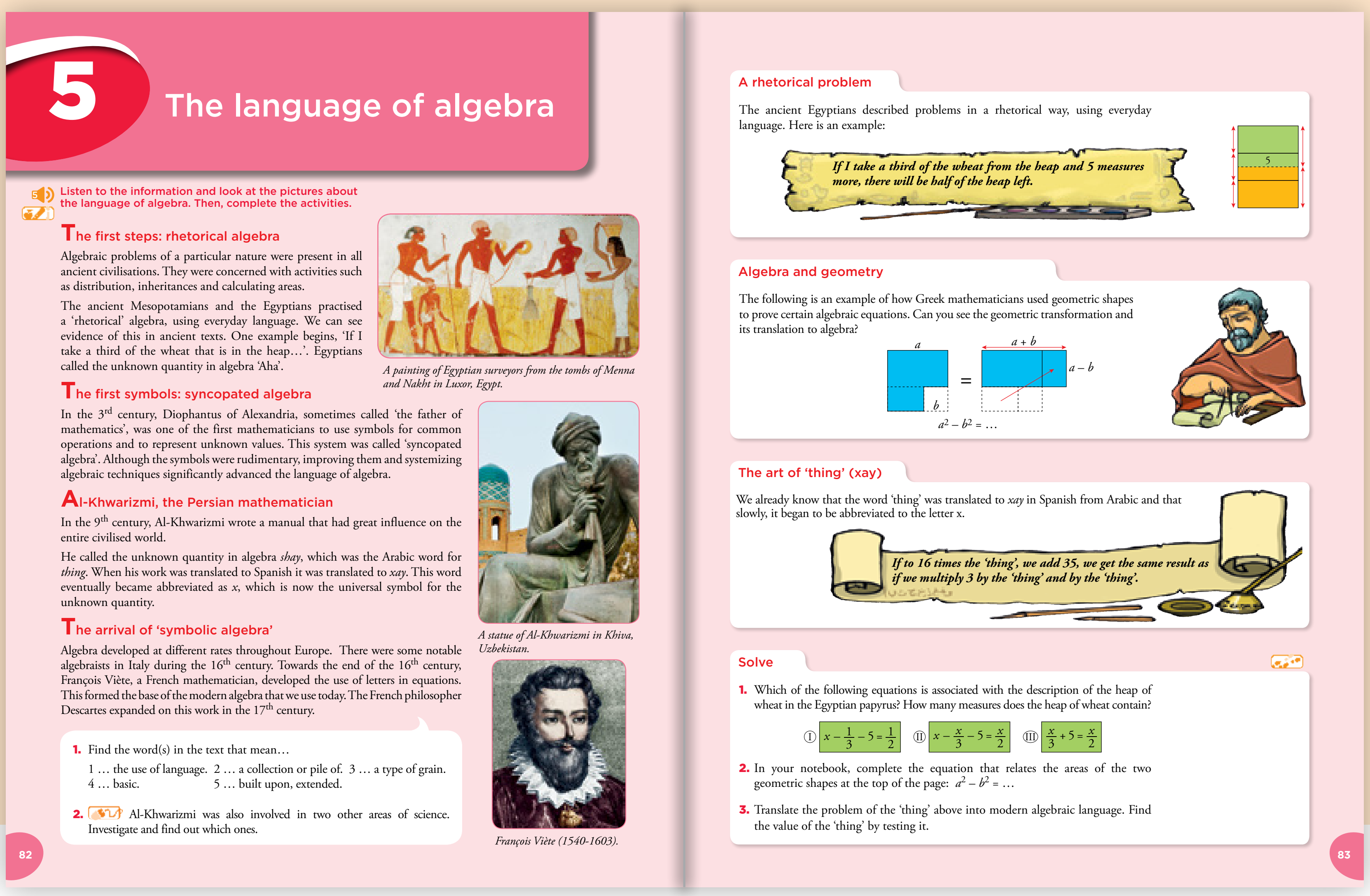
The language of algebra

Unit presentation

- We begin the study of algebra by reviewing and expanding on procedures learned in previous years.
- The main challenge students face is using letters as symbols that represent abstract situations, which is, at the same time, the main advantage that algebra gives us: using a letter to represent values, so we can manage the values more easily.
- The first section justifies why we need the language of algebra, reviews the meaning of certain terms and discusses the difference between identity and equation.
- The following pages focus on relevant definitions, the terminology associated with monomials and polynomials, their operations and their properties.
- The mastery of basic operations, sum and product, between monomials and polynomials, including extracting common factors, as well as the recognition of identities, will convince students that the transformation of complex expressions into other identical expressions, which are simpler, is one of the most effective methods in mathematics.
- We will study quotients of polynomials and Ruffini's rule. Its use for the transformation of a polynomial into factors, along with extracting the common factor and notable identities will be applied to the simplification of algebraic fractions. This part tends to be particularly difficult for students, so it is important that you choose the activities students will solve carefully. Remember that they will continue to expand their knowledge about these particular concepts in the following academic year.
- Throughout the unit, we will reinforce the learning of certain operations that appear frequently when solving equations (reducing to a common denominator or extracting common factors). These operations will be really useful during the following unit.

Unit outline





5 The language of algebra

Listen to the information and look at the pictures about the language of algebra. Then, complete the activities.

The first steps: rhetorical algebra

Algebraic problems of a particular nature were present in all ancient civilisations. They were concerned with activities such as distribution, inheritances and calculating areas.

The ancient Mesopotamians and the Egyptians practised a 'rhetorical' algebra, using everyday language. We can see evidence of this in ancient texts. One example begins, 'If I take a third of the wheat that is in the heap…'. Egyptians called the unknown quantity in algebra 'Aha'.

A painting of Egyptian surveyors from the tombs of Menna and Nakht in Luxor, Egypt.

The first symbols: syncopated algebra

In the 3rd century, Diophantus of Alexandria, sometimes called 'the father of mathematics', was one of the first mathematicians to use symbols for common operations and to represent unknown values. This system was called 'syncopated algebra'. Although the symbols were rudimentary, improving them and systemizing algebraic techniques significantly advanced the language of algebra.

Al-Khwarizmi, the Persian mathematician

In the 9th century, Al-Khwarizmi wrote a manual that had great influence on the entire civilised world.

He called the unknown quantity in algebra *shay*, which was the Arabic word for *thing*. When his work was translated to Spanish it was translated to *xay*. This word eventually became abbreviated as *x*, which is now the universal symbol for the unknown quantity.

A statue of Al-Khwarizmi in Khiva, Uzbekistan.

The arrival of 'symbolic algebra'

Algebra developed at different rates throughout Europe. There were some notable algebraists in Italy during the 16th century. Towards the end of the 16th century, François Viète, a French mathematician, developed the use of letters in equations. This formed the base of the modern algebra that we use today. The French philosopher Descartes expanded on this work in the 17th century.

François Viète (1540-1603).

1. Find the word(s) in the text that mean…

1 … the use of language. 2 … a collection or pile of. 3 … a type of grain.
4 … basic. 5 … built upon, extended.

2. Al-Khwarizmi was also involved in two other areas of science. Investigate and find out which ones.

A rhetorical problem

The ancient Egyptians described problems in a rhetorical way, using everyday language. Here is an example:

If I take a third of the wheat from the heap and 5 measures more, there will be half of the heap left.

Algebra and geometry

The following is an example of how Greek mathematicians used geometric shapes to prove certain algebraic equations. Can you see the geometric transformation and its translation to algebra?

$a^2 - b^2 = \dots$

The art of 'thing' (xay)

We already know that the word 'thing' was translated to *xay* in Spanish from Arabic and that slowly, it began to be abbreviated to the letter x.

If to 16 times the 'thing', we add 35, we get the same result as if we multiply 3 by the 'thing' and by the 'thing'.

Solve

1. Which of the following equations is associated with the description of the heap of wheat in the Egyptian papyrus? How many measures does the heap of wheat contain?

Ⓘ $x - \frac{1}{3} - 5 = \frac{1}{2}$ Ⓘⅰ (II) $x - \frac{x}{3} - 5 = \frac{x}{2}$ (III) $\frac{x}{3} + 5 = \frac{x}{2}$

2. In your notebook, complete the equation that relates the areas of the two geometric shapes at the top of the page: $a^2 - b^2 = \dots$

3. Translate the problem of the 'thing' above into modern algebraic language. Find the value of the 'thing' by testing it.

Starting the unit

- The unit starts with four short texts that talk about important milestones in the development and advancement of algebra. The aim of the texts is to make students understand the many steps needed until we finally found the nomenclature we use today. Before listening to the audio, divide the class into groups of four. Each group will be in charge of reading and summarising one of the four texts on the page.
- By reading the texts, students will become aware of the different stages in this long, historical process:
  - Rhetorical algebra: no abbreviations or special symbols. Natural language is used.
  - Geometrical algebra: uses geometric elements.
  - Syncopated algebra: certain technical terms and abbreviations are used.
  - Arabic algebra: the unknown quantity was referred to as the 'thing'.
  - Symbolic algebra: this algebra was much more similar to the one we use today.
- It might be interesting for students to learn that one of the reasons algebra developed in the Arabic world was that they had to solve very complex inheritance problems that were a result of their polygamous society.

Detecting previous knowledge

After students have listened to the audio and completed the related activities, they should make a list of as many circumstances in daily life they might confront where they would need algebra to resolve the problem. Once their lists are created, discuss their ideas as a class.

Answers to activities (page 82)

1 1 rhetorical; 2 heap; 3 wheat; 4 rudimentary; 5 expanded.

2 Astronomy and Geography.

ICT

Students could look for additional information about the Arab mathematician Al-Jwarizmi . How did his work reach the West?

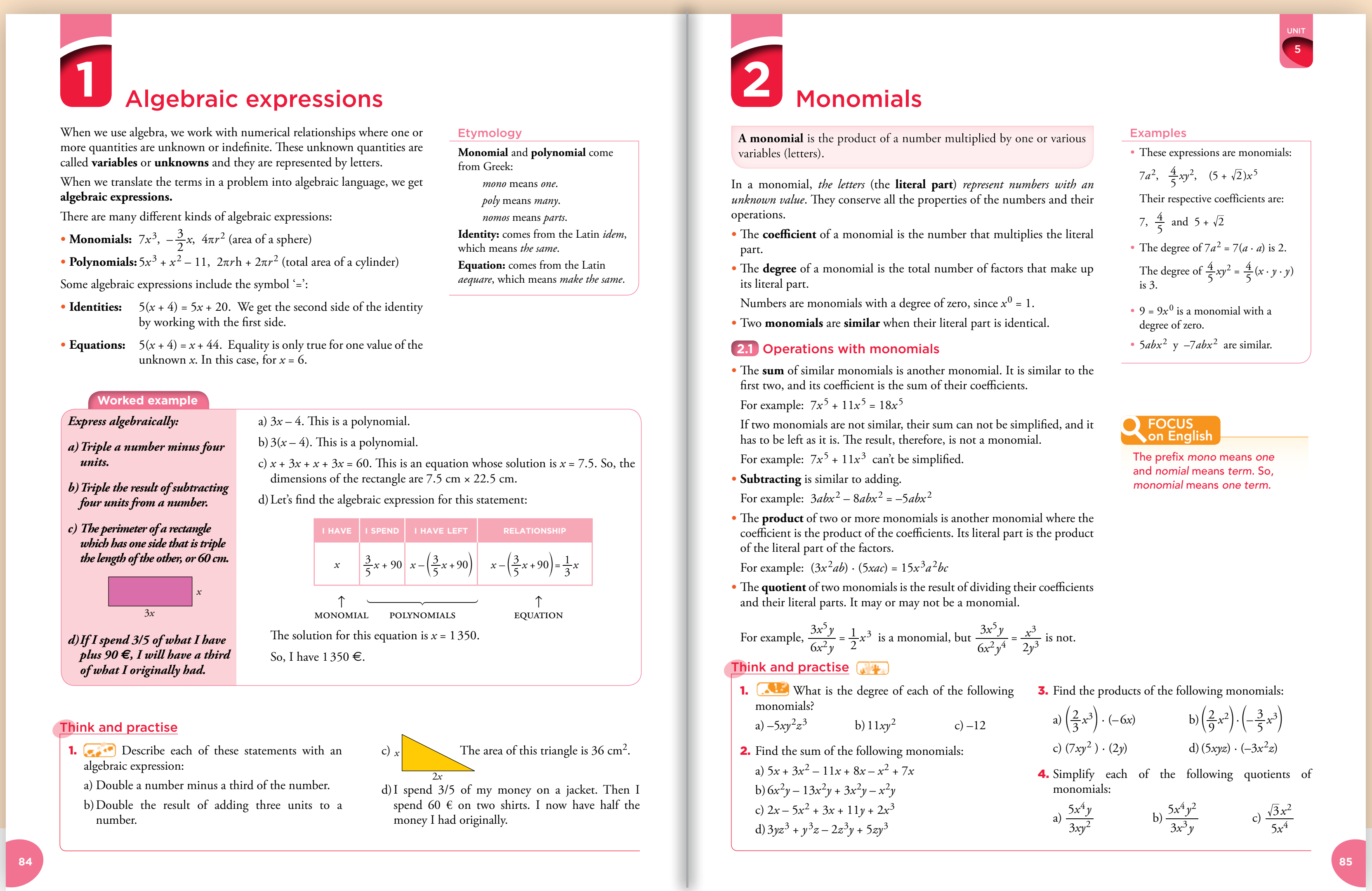
Answers to 'Solve'

1 Equation II. The heap of wheat contains 30 *measures*.

2 $a^2 - b^2 = (a + b) \cdot (a - b)$

3 $16x + 35 = 3x^2$. The *thing* has a value of 7.

Notes



1 Algebraic expressions

When we use algebra, we work with numerical relationships where one or more quantities are unknown or indefinite. These unknown quantities are called **variables** or **unknowns** and they are represented by letters.

When we translate the terms in a problem into algebraic language, we get **algebraic expressions.**

There are many different kinds of algebraic expressions:

- **Monomials:** $7x^3$, $-\frac{3}{2}x$, $4\pi r^2$ (area of a sphere)
- **Polynomials:** $5x^3 + x^2 - 11$, $2\pi r h + 2\pi r^2$ (total area of a cylinder)

Some algebraic expressions include the symbol '=':

- **Identities:** $5(x + 4) = 5x + 20$. We get the second side of the identity by working with the first side.
- **Equations:** $5(x + 4) = x + 44$. Equality is only true for one value of the unknown x. In this case, for $x = 6$.

Etymology

Monomial and **polynomial** come from Greek:

mono means *one*.
poly means *many*.
nomos means *parts*.

Identity: comes from the Latin *idem*, which means *the same*.

Equation: comes from the Latin *aequare*, which means *make the same*.

Worked example

Express algebraically:

a) Triple a number minus four units.

b) Triple the result of subtracting four units from a number.

c) The perimeter of a rectangle which has one side that is triple the length of the other, or 60 cm.

x
$3x$

d) If I spend 3/5 of what I have plus 90 €, I will have a third of what I originally had.

a) $3x - 4$. This is a polynomial.

b) $3(x - 4)$. This is a polynomial.

c) $x + 3x + x + 3x = 60$. This is an equation whose solution is $x = 7.5$. So, the dimensions of the rectangle are 7.5 cm × 22.5 cm.

d) Let's find the algebraic expression for this statement:

I HAVE	I SPEND	I HAVE LEFT	RELATIONSHIP
x	$\frac{3}{5}x + 90$	$x - \left(\frac{3}{5}x + 90\right)$	$x - \left(\frac{3}{5}x + 90\right) = \frac{1}{3}x$

↑ MONOMIAL — POLYNOMIALS — ↑ EQUATION

The solution for this equation is $x = 1\,350$.

So, I have 1 350 €.

Think and practise

1. Describe each of these statements with an algebraic expression:

a) Double a number minus a third of the number.

b) Double the result of adding three units to a number.

c) x $2x$ The area of this triangle is 36 cm^2.

d) I spend 3/5 of my money on a jacket. Then I spend 60 € on two shirts. I now have half the money I had originally.

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UNIT 5

2 Monomials

A monomial is the product of a number multiplied by one or various variables (letters).

In a monomial, *the letters* (the **literal part**) *represent numbers with an unknown value*. They conserve all the properties of the numbers and their operations.

- The **coefficient** of a monomial is the number that multiplies the literal part.
- The **degree** of a monomial is the total number of factors that make up its literal part.

  Numbers are monomials with a degree of zero, since $x^0 = 1$.
- Two **monomials** are **similar** when their literal part is identical.

Examples

- These expressions are monomials:

  $7a^2$, $\frac{4}{5}xy^2$, $(5 + \sqrt{2})x^5$

  Their respective coefficients are:

  7, $\frac{4}{5}$ and $5 + \sqrt{2}$
- The degree of $7a^2 = 7(a \cdot a)$ is 2.

  The degree of $\frac{4}{5}xy^2 = \frac{4}{5}(x \cdot y \cdot y)$ is 3.
- $9 = 9x^0$ is a monomial with a degree of zero.
- $5abx^2$ y $-7abx^2$ are similar.

2.1 Operations with monomials

- The **sum** of similar monomials is another monomial. It is similar to the first two, and its coefficient is the sum of their coefficients.

  For example: $7x^5 + 11x^5 = 18x^5$

  If two monomials are not similar, their sum can not be simplified, and it has to be left as it is. The result, therefore, is not a monomial.

  For example: $7x^5 + 11x^3$ can't be simplified.
- **Subtracting** is similar to adding.

  For example: $3abx^2 - 8abx^2 = -5abx^2$
- The **product** of two or more monomials is another monomial where the coefficient is the product of the coefficients. Its literal part is the product of the literal part of the factors.

  For example: $(3x^2ab) \cdot (5xac) = 15x^3a^2bc$
- The **quotient** of two monomials is the result of dividing their coefficients and their literal parts. It may or may not be a monomial.

  For example, $\frac{3x^5y}{6x^2y} = \frac{1}{2}x^3$ is a monomial, but $\frac{3x^5y}{6x^2y^4} = \frac{x^3}{2y^3}$ is not.

FOCUS on English

The prefix *mono* means *one* and *nomial* means *term*. So, *monomial* means *one term.*

Think and practise

1. What is the degree of each of the following monomials?

a) $-5xy^2z^3$ b) $11xy^2$ c) -12

2. Find the sum of the following monomials:

a) $5x + 3x^2 - 11x + 8x - x^2 + 7x$

b) $6x^2y - 13x^2y + 3x^2y - x^2y$

c) $2x - 5x^2 + 3x + 11y + 2x^3$

d) $3yz^3 + y^3z - 2z^3y + 5zy^3$

3. Find the products of the following monomials:

a) $\left(\frac{2}{3}x^3\right) \cdot (-6x)$ b) $\left(\frac{2}{9}x^2\right) \cdot \left(-\frac{3}{5}x^3\right)$

c) $(7xy^2) \cdot (2y)$ d) $(5xyz) \cdot (-3x^2z)$

4. Simplify each of the following quotients of monomials:

a) $\frac{5x^4y}{3xy^2}$ b) $\frac{5x^4y^2}{3x^3y}$ c) $\frac{\sqrt{3}x^2}{5x^4}$

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Suggestions

- In section 1, we remind students of basic algebraic terminology and we show them its main use: translating a question or a property into a symbolic language.
- Making sure students use algebraic expressions effectively is a very important part of the students' learning at this level. Students also need to learn about specific situations that are described by algebraic expressions in a symbolic way. This is why students should first learn to associate a range of questions to their corresponding algebraic expressions, so that they then are able to obtain algebraic expressions that relate to questions independently.
- If appropriate, you might want to explore the meaning of letters further:
  - Unknown: a letter that represents an unknown quantity but can be calculated.
  - Variable: letters than can have any value.
- In section 2, we remind students of the definition of monomial as well as the vocabulary related to basic operations: sum, product and quotient of monomials. This operation can be presented to students as an extension of arithmetic operations: extraction of common factors and product or quotient of two powers with the same base.
- We can also check that, when we carry out a sum or product of monomials ($5x^2 + 3x^2 = 8x^2$ or $3x \cdot 2x^2 = 6x^3$), the value of the elements is the same on both sides, regardless of the value that we assign to the letters.

Focus on English

FOCUS on English

- Ask students to think of other words that begin with the prefix *mono-*. Examples: *monologue, monogamous, monorail, monoparental.*

Cooperative learning

We suggest the following methodology to develop students' ability to carry out algebraic operations.

- Students work in small groups (two or three students per group).
- They solve a series of expressions individually and then they compare processes and answers.
- If students have different procedures or answers, they should work together to spot the mistakes. If they are unsure or they cannot agree, the teacher should intervene to clarify any misconceptions.

Answers to 'Think and practise' (page 84)

1 a) $2x - \frac{1}{3}x$ b) $2(x - 3)$

c) $\frac{2x \cdot x}{2} = 36$ d) $x - \left(\frac{3}{5}x + 60\right) = \frac{1}{2}x$

Answers to 'Think and practise' (page 85)

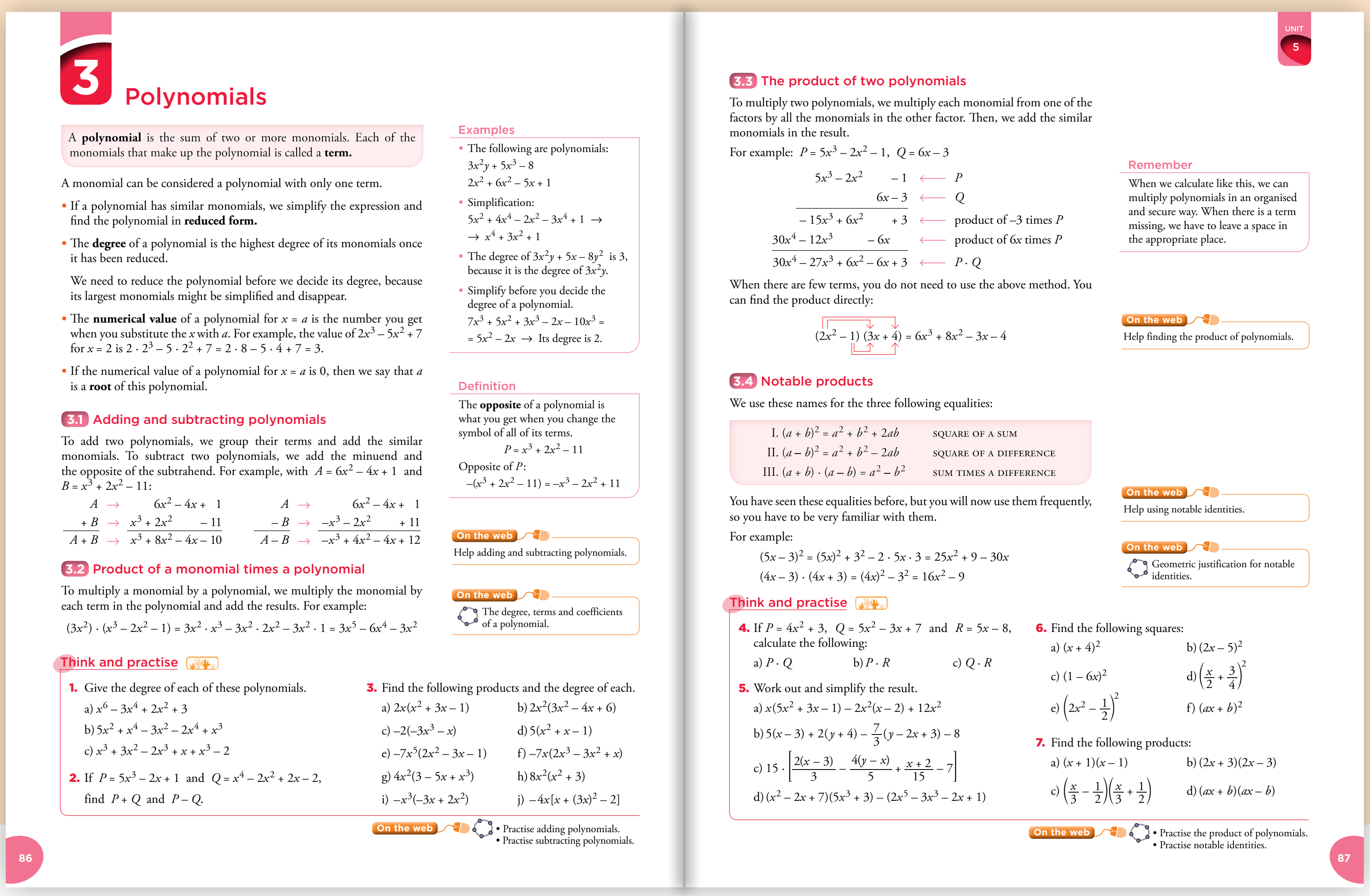
1 a) Degree 6 b) Degree 3 c) Degree 0

2 a) $9x + 2x^2$ b) $-5x^2y$

c) $5x - 5x^2 + 2x^3 + 11y$ d) $4z^3 + 6y^3z$

3 a) $-4x^4$ b) $-2x^5/15$ c) $14xy^3$ d) $-15x^3yz^2$

4 a) $\frac{5x^3}{3y}$ b) $\frac{5xy}{3}$ c) $\frac{\sqrt{3}}{5x^2}$



3 Polynomials

A **polynomial** is the sum of two or more monomials. Each of the monomials that make up the polynomial is called a **term.**

A monomial can be considered a polynomial with only one term.

- If a polynomial has similar monomials, we simplify the expression and find the polynomial in **reduced form.**
- The **degree** of a polynomial is the highest degree of its monomials once it has been reduced.

  We need to reduce the polynomial before we decide its degree, because its largest monomials might be simplified and disappear.
- The **numerical value** of a polynomial for $x = a$ is the number you get when you substitute the x with a. For example, the value of $2x^3 - 5x^2 + 7$ for $x = 2$ is $2 \cdot 2^3 - 5 \cdot 2^2 + 7 = 2 \cdot 8 - 5 \cdot 4 + 7 = 3$.
- If the numerical value of a polynomial for $x = a$ is 0, then we say that a is a **root** of this polynomial.

3.1 Adding and subtracting polynomials

To add two polynomials, we group their terms and add the similar monomials. To subtract two polynomials, we add the minuend and the opposite of the subtrahend. For example, with $A = 6x^2 - 4x + 1$ and $B = x^3 + 2x^2 - 11$:

$$\begin{array}{rcrrrr} A & \rightarrow & & 6x^2 & -\,4x & +\,1 \\ +\,B & \rightarrow & x^3 & +\,2x^2 & & -\,11 \\ \hline A + B & \rightarrow & x^3 & +\,8x^2 & -\,4x & -\,10 \end{array} \qquad \begin{array}{rcrrrr} A & \rightarrow & & 6x^2 & -\,4x & +\,1 \\ -\,B & \rightarrow & -x^3 & -\,2x^2 & & +\,11 \\ \hline A - B & \rightarrow & -x^3 & +\,4x^2 & -\,4x & +\,12 \end{array}$$

3.2 Product of a monomial times a polynomial

To multiply a monomial by a polynomial, we multiply the monomial by each term in the polynomial and add the results. For example:

$$(3x^2) \cdot (x^3 - 2x^2 - 1) = 3x^2 \cdot x^3 - 3x^2 \cdot 2x^2 - 3x^2 \cdot 1 = 3x^5 - 6x^4 - 3x^2$$

Examples

- The following are polynomials:
  $3x^2y + 5x^3 - 8$
  $2x^2 + 6x^2 - 5x + 1$
- Simplification:
  $5x^2 + 4x^4 - 2x^2 - 3x^4 + 1 \rightarrow$
  $\rightarrow x^4 + 3x^2 + 1$
- The degree of $3x^2y + 5x - 8y^2$ is 3, because it is the degree of $3x^2y$.
- Simplify before you decide the degree of a polynomial.
  $7x^3 + 5x^2 + 3x^3 - 2x - 10x^3 =$
  $= 5x^2 - 2x \rightarrow$ Its degree is 2.

Definition

The **opposite** of a polynomial is what you get when you change the symbol of all of its terms.

$$P = x^3 + 2x^2 - 11$$

Opposite of P:

$$-(x^3 + 2x^2 - 11) = -x^3 - 2x^2 + 11$$

On the web
Help adding and subtracting polynomials.

On the web
The degree, terms and coefficients of a polynomial.

Think and practise

1. Give the degree of each of these polynomials.
a) $x^6 - 3x^4 + 2x^2 + 3$
b) $5x^2 + x^4 - 3x^2 - 2x^4 + x^3$
c) $x^3 + 3x^2 - 2x^3 + x + x^3 - 2$

2. If $P = 5x^3 - 2x + 1$ and $Q = x^4 - 2x^2 + 2x - 2$, find $P + Q$ and $P - Q$.

3. Find the following products and the degree of each.
a) $2x(x^2 + 3x - 1)$ b) $2x^2(3x^2 - 4x + 6)$
c) $-2(-3x^3 - x)$ d) $5(x^2 + x - 1)$
e) $-7x^5(2x^2 - 3x - 1)$ f) $-7x(2x^3 - 3x^2 + x)$
g) $4x^2(3 - 5x + x^3)$ h) $8x^2(x^2 + 3)$
i) $-x^3(-3x + 2x^2)$ j) $-4x[x + (3x)^2 - 2]$

On the web
- Practise adding polynomials.
- Practise subtracting polynomials.

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UNIT 5

3.3 The product of two polynomials

To multiply two polynomials, we multiply each monomial from one of the factors by all the monomials in the other factor. Then, we add the similar monomials in the result.

For example: $P = 5x^3 - 2x^2 - 1$, $Q = 6x - 3$

$$\begin{array}{rrrrrcl} & 5x^3 & -\,2x^2 & & -\,1 & \longleftarrow & P \\ & & & 6x & -\,3 & \longleftarrow & Q \\ \hline & -\,15x^3 & +\,6x^2 & & +\,3 & \longleftarrow & \text{product of } -3 \text{ times } P \\ 30x^4 & -\,12x^3 & & -\,6x & & \longleftarrow & \text{product of } 6x \text{ times } P \\ \hline 30x^4 & -\,27x^3 & +\,6x^2 & -\,6x & +\,3 & \longleftarrow & P \cdot Q \end{array}$$

When there are few terms, you do not need to use the above method. You can find the product directly:

$$(2x^2 - 1)\,(3x + 4) = 6x^3 + 8x^2 - 3x - 4$$

3.4 Notable products

We use these names for the three following equalities:

I. $(a + b)^2 = a^2 + b^2 + 2ab$ SQUARE OF A SUM
II. $(a - b)^2 = a^2 + b^2 - 2ab$ SQUARE OF A DIFFERENCE
III. $(a + b) \cdot (a - b) = a^2 - b^2$ SUM TIMES A DIFFERENCE

You have seen these equalities before, but you will now use them frequently, so you have to be very familiar with them.

For example:

$$(5x - 3)^2 = (5x)^2 + 3^2 - 2 \cdot 5x \cdot 3 = 25x^2 + 9 - 30x$$
$$(4x - 3) \cdot (4x + 3) = (4x)^2 - 3^2 = 16x^2 - 9$$

Remember

When we calculate like this, we can multiply polynomials in an organised and secure way. When there is a term missing, we have to leave a space in the appropriate place.

On the web
Help finding the product of polynomials.

On the web
Help using notable identities.

On the web
Geometric justification for notable identities.

Think and practise

4. If $P = 4x^2 + 3$, $Q = 5x^2 - 3x + 7$ and $R = 5x - 8$, calculate the following:
a) $P \cdot Q$ b) $P \cdot R$ c) $Q \cdot R$

5. Work out and simplify the result.
a) $x(5x^2 + 3x - 1) - 2x^2(x - 2) + 12x^2$
b) $5(x - 3) + 2(y + 4) - \frac{7}{3}(y - 2x + 3) - 8$
c) $15 \cdot \left[\frac{2(x-3)}{3} - \frac{4(y-x)}{5} + \frac{x+2}{15} - 7\right]$
d) $(x^2 - 2x + 7)(5x^3 + 3) - (2x^5 - 3x^3 - 2x + 1)$

6. Find the following squares:
a) $(x + 4)^2$ b) $(2x - 5)^2$
c) $(1 - 6x)^2$ d) $\left(\frac{x}{2} + \frac{3}{4}\right)^2$
e) $\left(2x^2 - \frac{1}{2}\right)^2$ f) $(ax + b)^2$

7. Find the following products:
a) $(x + 1)(x - 1)$ b) $(2x + 3)(2x - 3)$
c) $\left(\frac{x}{3} - \frac{1}{2}\right)\left(\frac{x}{3} + \frac{1}{2}\right)$ d) $(ax + b)(ax - b)$

On the web
- Practise the product of polynomials.
- Practise notable identities.

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Suggestions

- The sum of monomials or the product of a monomial and a polynomial, or the product of two polynomials, all involve working with monomials. It is useful to place polynomials one below the other to work in an organised way and to avoid mistakes. This help us to group similar monomials together so that we can simplify them. It is up to the teacher to decide when it is appropriate for students to write polynomials on one line only, so the required operations are carried out in a successive way.
- Although students know notable identities from previous years, often, a great number of them are not fully confident when using them. So, it is important to justify their development as the product of binomials, as well as practise with as many examples as possible.

Cooperative learning

We suggest the following methodology to develop students' ability to carry out algebraic operations.

- Students work in small groups (two or three students per group).
- They solve a series of expressions individually and then they compare processes and answers.

Answers to 'Think and practise'

1 a) Degree 6 b) Degree 4 c) Degree 2

2 $P + Q = x^4 + 5x^3 - 2x^2 - 1$; $P - Q = -x^4 + 5x^3 + 2x^2 - 4x + 3$

3 a) $2x^3 + 6x^2 - 2x$. Degree 3 b) $6x^4 - 8x^3 + 12x^2$. Degree 4
c) $6x^3 + 2x$. Degree 3 d) $5x^2 + 5x - 5$. Degree 2
e) $-14x^7 + 21x^6 + 7x^5$. Degree 7 f) $-14x^4 + 21x^3 - 7x^2$. Degree 4
g) $12x^2 - 20x^3 + 4x^5$. Degree 5 h) $8x^4 + 24x^2$. Degree 4
i) $3x^4 - 2x^5$. Degree 5 j) $-4x^2 - 36x^3 + 8x$. Degree 3

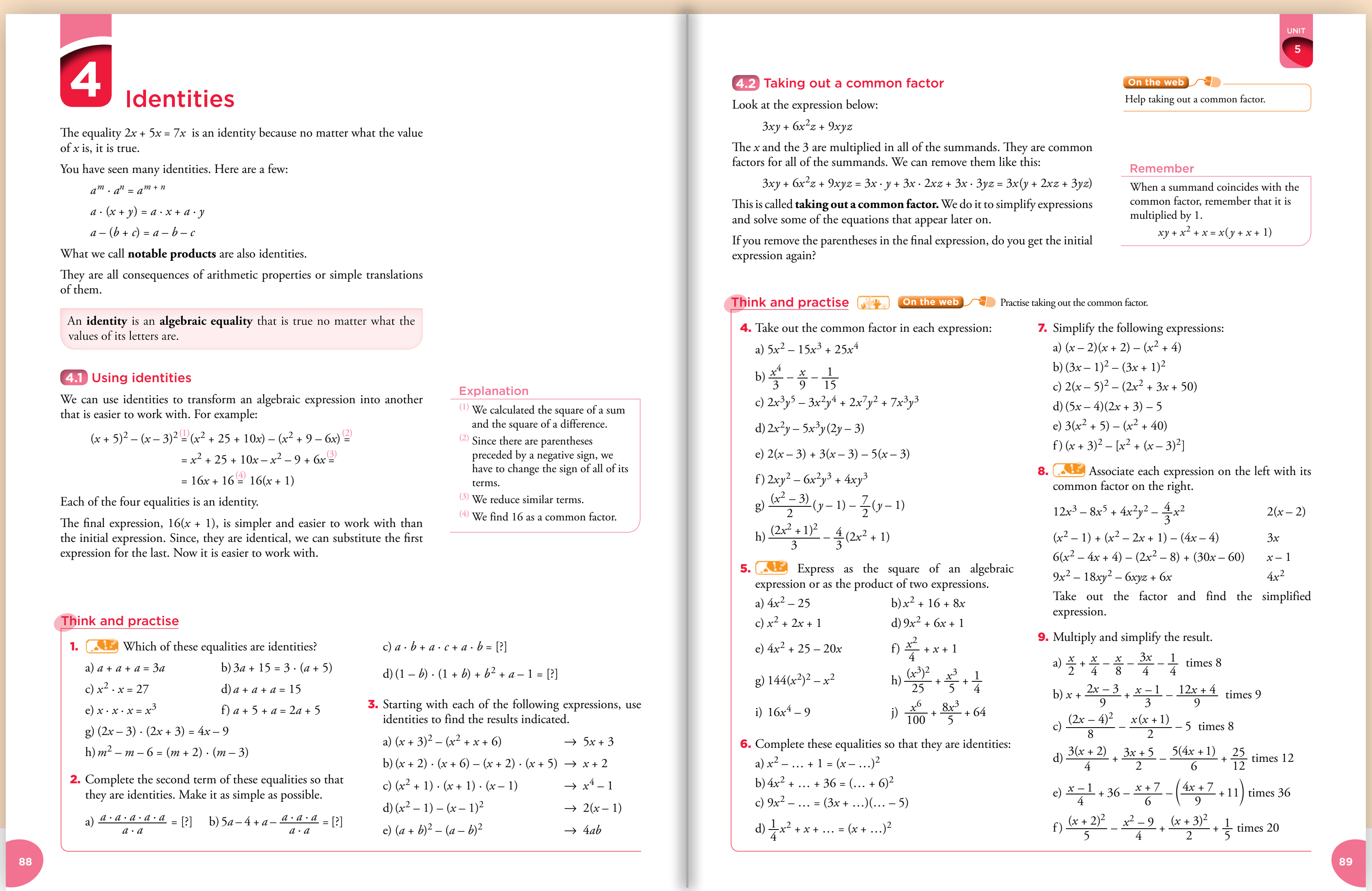
4 a) $P \cdot Q = 20x^4 - 12x^3 + 43x^2 - 9x + 21$
b) $P \cdot R = 20x^3 - 32x^2 + 15x - 24$
c) $Q \cdot R = 25x^3 - 55x^2 + 59x - 56$

5 a) $3x^3 + 19x^2 - x$ b) $\frac{29}{3}x - \frac{1}{3}y - 22$
c) $23x - 12y - 133$ d) $3x^5 - 10x^4 + 38x^3 + 3x^2 - 4x + 20$

6 a) $x^2 + 16 + 8x$ b) $4x^2 + 25 - 20x$ c) $1 + 36x^2 - 12x$
d) $\frac{x^2}{4} + \frac{x^2}{16} + \frac{3x}{4} = \frac{1}{16}(4x^2 + 9 + 12x)$
e) $4x^4 + \frac{1}{4} - 2x^2 = \frac{1}{4}(16x^4 + 1 - 8x^2)$
f) $a^2x^2 + b^2 + 2abx$

7 a) $x^2 - 1$ b) $4x^2 - 9$
c) $\frac{x^2}{9} - \frac{1}{4}$ d) $a^2x^2 - b^2$

Notes



4 Identities

The equality $2x + 5x = 7x$ is an identity because no matter what the value of x is, it is true.

You have seen many identities. Here are a few:

$$a^m \cdot a^n = a^{m+n}$$
$$a \cdot (x + y) = a \cdot x + a \cdot y$$
$$a - (b + c) = a - b - c$$

What we call **notable products** are also identities.

They are all consequences of arithmetic properties or simple translations of them.

An **identity** is an **algebraic equality** that is true no matter what the values of its letters are.

4.1 Using identities

We can use identities to transform an algebraic expression into another that is easier to work with. For example:

$$(x + 5)^2 - (x - 3)^2 \overset{(1)}{=} (x^2 + 25 + 10x) - (x^2 + 9 - 6x) \overset{(2)}{=}$$
$$= x^2 + 25 + 10x - x^2 - 9 + 6x \overset{(3)}{=}$$
$$= 16x + 16 \overset{(4)}{=} 16(x + 1)$$

Each of the four equalities is an identity.

The final expression, $16(x + 1)$, is simpler and easier to work with than the initial expression. Since, they are identical, we can substitute the first expression for the last. Now it is easier to work with.

Explanation

(1) We calculated the square of a sum and the square of a difference.

(2) Since there are parentheses preceded by a negative sign, we have to change the sign of all of its terms.

(3) We reduce similar terms.

(4) We find 16 as a common factor.

Think and practise

1. Which of these equalities are identities?

a) $a + a + a = 3a$
b) $3a + 15 = 3 \cdot (a + 5)$
c) $x^2 \cdot x = 27$
d) $a + a + a = 15$
e) $x \cdot x \cdot x = x^3$
f) $a + 5 + a = 2a + 5$
g) $(2x - 3) \cdot (2x + 3) = 4x - 9$
h) $m^2 - m - 6 = (m + 2) \cdot (m - 3)$

2. Complete the second term of these equalities so that they are identities. Make it as simple as possible.

a) $\frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a}$ = [?]
b) $5a - 4 + a - \frac{a \cdot a \cdot a}{a \cdot a}$ = [?]
c) $a \cdot b + a \cdot c + a \cdot b$ = [?]
d) $(1 - b) \cdot (1 + b) + b^2 + a - 1$ = [?]

3. Starting with each of the following expressions, use identities to find the results indicated.

a) $(x + 3)^2 - (x^2 + x + 6) \rightarrow 5x + 3$
b) $(x + 2) \cdot (x + 6) - (x + 2) \cdot (x + 5) \rightarrow x + 2$
c) $(x^2 + 1) \cdot (x + 1) \cdot (x - 1) \rightarrow x^4 - 1$
d) $(x^2 - 1) - (x - 1)^2 \rightarrow 2(x - 1)$
e) $(a + b)^2 - (a - b)^2 \rightarrow 4ab$

4.2 Taking out a common factor

Look at the expression below:

$$3xy + 6x^2z + 9xyz$$

The x and the 3 are multiplied in all of the summands. They are common factors for all of the summands. We can remove them like this:

$$3xy + 6x^2z + 9xyz = 3x \cdot y + 3x \cdot 2xz + 3x \cdot 3yz = 3x(y + 2xz + 3yz)$$

This is called **taking out a common factor.** We do it to simplify expressions and solve some of the equations that appear later on.

If you remove the parentheses in the final expression, do you get the initial expression again?

On the web
Help taking out a common factor.

Remember

When a summand coincides with the common factor, remember that it is multiplied by 1.

$$xy + x^2 + x = x(y + x + 1)$$

Think and practise

On the web Practise taking out the common factor.

4. Take out the common factor in each expression:

a) $5x^2 - 15x^3 + 25x^4$
b) $\frac{x^4}{3} - \frac{x}{9} - \frac{1}{15}$
c) $2x^3y^5 - 3x^2y^4 + 2x^7y^2 + 7x^3y^3$
d) $2x^2y - 5x^3y(2y - 3)$
e) $2(x - 3) + 3(x - 3) - 5(x - 3)$
f) $2xy^2 - 6x^2y^3 + 4xy^3$
g) $\frac{(x^2 - 3)}{2}(y - 1) - \frac{7}{2}(y - 1)$
h) $\frac{(2x^2 + 1)^2}{3} - \frac{4}{3}(2x^2 + 1)$

5. Express as the square of an algebraic expression or as the product of two expressions.

a) $4x^2 - 25$
b) $x^2 + 16 + 8x$
c) $x^2 + 2x + 1$
d) $9x^2 + 6x + 1$
e) $4x^2 + 25 - 20x$
f) $\frac{x^2}{4} + x + 1$
g) $144(x^2)^2 - x^2$
h) $\frac{(x^3)^2}{25} + \frac{x^3}{5} + \frac{1}{4}$
i) $16x^4 - 9$
j) $\frac{x^6}{100} + \frac{8x^3}{5} + 64$

6. Complete these equalities so that they are identities:

a) $x^2 - \ldots + 1 = (x - \ldots)^2$
b) $4x^2 + \ldots + 36 = (\ldots + 6)^2$
c) $9x^2 - \ldots = (3x + \ldots)(\ldots - 5)$
d) $\frac{1}{4}x^2 + x + \ldots = (x + \ldots)^2$

7. Simplify the following expressions:

a) $(x - 2)(x + 2) - (x^2 + 4)$
b) $(3x - 1)^2 - (3x + 1)^2$
c) $2(x - 5)^2 - (2x^2 + 3x + 50)$
d) $(5x - 4)(2x + 3) - 5$
e) $3(x^2 + 5) - (x^2 + 40)$
f) $(x + 3)^2 - [x^2 + (x - 3)^2]$

8. Associate each expression on the left with its common factor on the right.

$12x^3 - 8x^5 + 4x^2y^2 - \frac{4}{3}x^2$	$2(x - 2)$
$(x^2 - 1) + (x^2 - 2x + 1) - (4x - 4)$	$3x$
$6(x^2 - 4x + 4) - (2x^2 - 8) + (30x - 60)$	$x - 1$
$9x^2 - 18xy^2 - 6xyz + 6x$	$4x^2$

Take out the factor and find the simplified expression.

9. Multiply and simplify the result.

a) $\frac{x}{2} + \frac{x}{4} - \frac{x}{8} - \frac{3x}{4} - \frac{1}{4}$ times 8
b) $x + \frac{2x - 3}{9} + \frac{x - 1}{3} - \frac{12x + 4}{9}$ times 9
c) $\frac{(2x - 4)^2}{8} - \frac{x(x + 1)}{2} - 5$ times 8
d) $\frac{3(x + 2)}{4} + \frac{3x + 5}{2} - \frac{5(4x + 1)}{6} + \frac{25}{12}$ times 12
e) $\frac{x - 1}{4} + 36 - \frac{x + 7}{6} - \left(\frac{4x + 7}{9} + 11\right)$ times 36
f) $\frac{(x + 2)^2}{5} - \frac{x^2 - 9}{4} + \frac{(x + 3)^2}{2} + \frac{1}{5}$ times 20

Suggestions

- The concept of identity (an algebraic equality that is true no matter what the value of the letters are), should be easy for students to understand. If appropriate, you might want to associate the concept of an equation with infinite solutions. This will be studied in the next unit.
- Once students are able to work with notable products, they need to also be familiar with the inverse step: recognising expressions that are the square of a binomial or the difference between squared monomials.
- Some students find removing a common factor difficult. Normally, this is because they have difficulty recognising the factors that can be extracted or with the division of monomials involved in the process. They may also struggle to identify the term that needs to be put inside the brackets when the quotient is the unit. As a first step, you may want to ask students to check the identity between both expressions.
- It is important to make students understand that the correct use of identities is key to solving equations, systems of equations and other algebraic processes. This is why we propose a range of activities in which the answer can be simplified once it has been obtained.

Cooperative learning

We suggest the following methodology to develop students' ability to carry out algebraic operations.

- Students work in small groups (two or three students per group).
- They solve a series of expressions individually and then they compare processes and answers.
- If students have different procedures or answers, they work together to spot the mistakes. If they are unsure or they cannot agree, the teacher should intervene to clarify any misconceptions.

Answers to 'Think and practise'

1 The following are equalities are identities: a), b), e), f) and h).

2 a) a^3 b) $5a - 4$ c) $2ab + ac$ d) a

3 Students check results.

4 a) $5x^2(1 - 3x + 5x^2)$

b) $\frac{1}{3}\left(x^4 - \frac{x}{3} - \frac{1}{5}\right)$

c) $x^2y^2(2xy^3 - 3y^2 + 2x^5 + 7xy)$

d) $x^2y(2 - 10xy + 15x)$

e) 0

f) $2xy^2(1 - 3xy + 2y)$

g) $(y - 1)\left(\frac{x^2 - 3 - 7}{2}\right) = (y - 1)\left(\frac{x^2}{2} - 5\right)$

h) $\frac{1}{3}(2x^2 + 1)(2x^2 - 3)$

5 a) $(2x + 5)(2x - 5)$ b) $(x + 4)^2$

c) $(x + 1)^2$ d) $(3x + 1)^2$

e) $(2x - 5)^2$ f) $\left(\frac{x}{2} + 1\right)^2$

g) $(12x^2 - x) \cdot (12x^2 + x)$ h) $\left(\frac{x^3}{5} + \frac{1}{2}\right)^2$

i) $(4x^2 - 3) \cdot (4x^2 + 3)$ j) $\left(\frac{x^3}{10} + 8\right)^2$

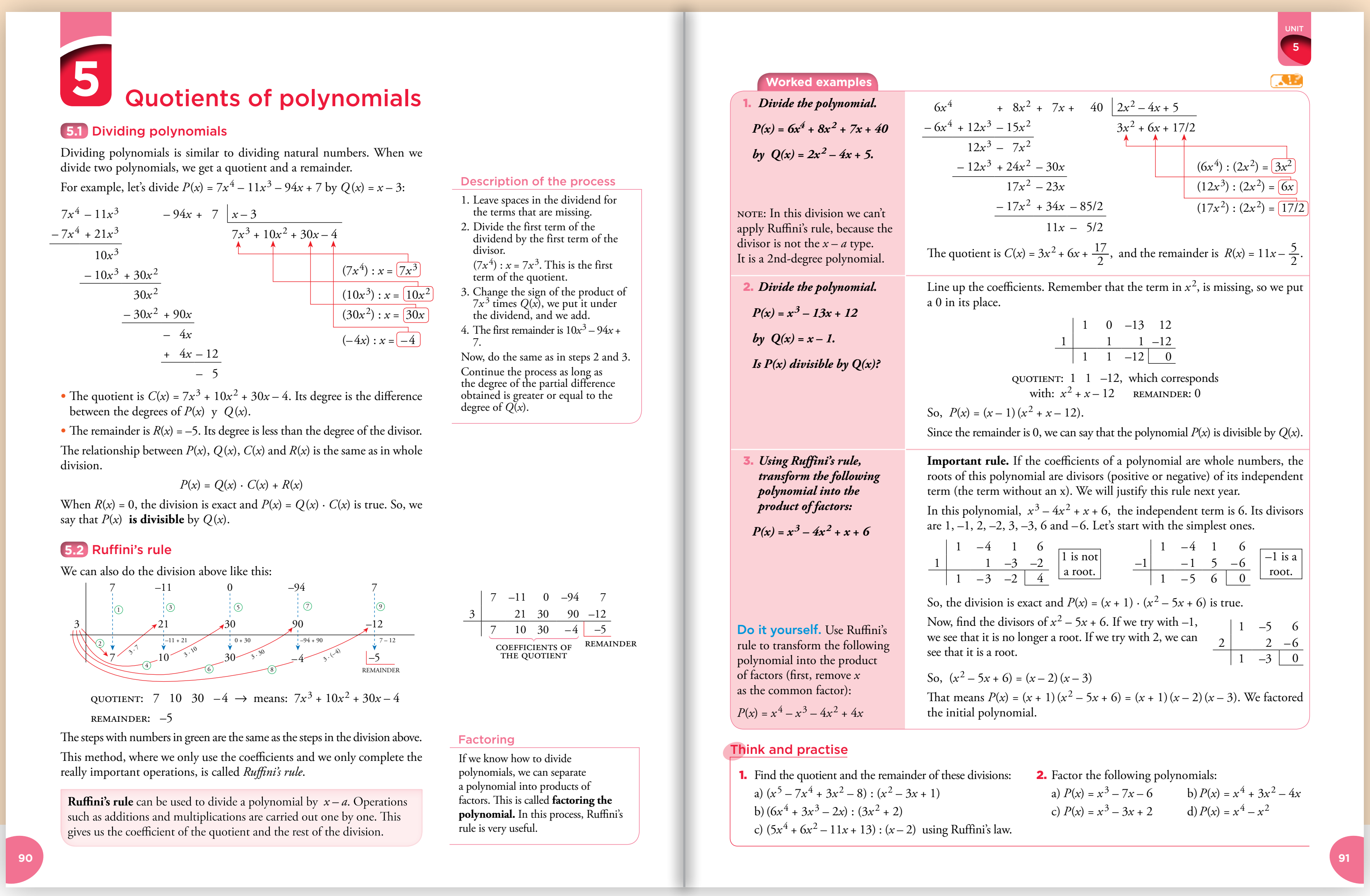
6 a) $x^2 - 2x + 1 = (x - 1)^2$

b) $4x^2 + 24x + 36 = (2x + 6)^2$

c) $9x^2 - 25 = (3x + 5) \cdot (3x - 5)$

d) $\frac{1}{4}x^2 + x + 1 = \left(x + \left(1 - \frac{1}{2}x\right)\right)^2$

7 to 9 Answers at the end of the unit..



UNIT 5

5 Quotients of polynomials

5.1 Dividing polynomials

Dividing polynomials is similar to dividing natural numbers. When we divide two polynomials, we get a quotient and a remainder.

For example, let's divide $P(x) = 7x^4 - 11x^3 - 94x + 7$ by $Q(x) = x - 3$:

$$\begin{array}{lllll|l}
7x^4 & -11x^3 & & -94x & +7 & x-3 \\ \hline
-7x^4 & +21x^3 & & & & 7x^3+10x^2+30x-4 \\ \hline
 & 10x^3 & & & & \\
 & -10x^3 & +30x^2 & & & \\ \hline
 & & 30x^2 & & & \\
 & & -30x^2 & +90x & & \\ \hline
 & & & -4x & & \\
 & & & +4x & -12 & \\ \hline
 & & & & -5 &
\end{array}$$

$(7x^4) : x = \boxed{7x^3}$

$(10x^3) : x = \boxed{10x^2}$

$(30x^2) : x = \boxed{30x}$

$(-4x) : x = \boxed{-4}$

- The quotient is $C(x) = 7x^3 + 10x^2 + 30x - 4$. Its degree is the difference between the degrees of $P(x)$ y $Q(x)$.
- The remainder is $R(x) = -5$. Its degree is less than the degree of the divisor.

The relationship between $P(x)$, $Q(x)$, $C(x)$ and $R(x)$ is the same as in whole division.

$$P(x) = Q(x) \cdot C(x) + R(x)$$

When $R(x) = 0$, the division is exact and $P(x) = Q(x) \cdot C(x)$ is true. So, we say that $P(x)$ **is divisible** by $Q(x)$.

Description of the process

1. Leave spaces in the dividend for the terms that are missing.
2. Divide the first term of the dividend by the first term of the divisor.
   $(7x^4) : x = 7x^3$. This is the first term of the quotient.
3. Change the sign of the product of $7x^3$ times $Q(x)$, we put it under the dividend, and we add.
4. The first remainder is $10x^3 - 94x + 7$.

Now, do the same as in steps 2 and 3.

Continue the process as long as the degree of the partial difference obtained is greater or equal to the degree of $Q(x)$.

5.2 Ruffini's rule

We can also do the division above like this:

$$\begin{array}{r|rrrrr}
 & 7 & -11 & 0 & -94 & 7 \\
3 & & 21 & 30 & 90 & -12 \\ \hline
 & 7 & 10 & 30 & -4 & -5
\end{array}$$

① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

$-11 + 21$ $0 + 30$ $-94 + 90$ $7 - 12$

$3 \cdot 7$ $3 \cdot 10$ $3 \cdot 30$ $3 \cdot (-4)$

REMAINDER

QUOTIENT: 7 10 30 –4 → means: $7x^3 + 10x^2 + 30x - 4$

REMAINDER: –5

$$\begin{array}{r|rrrr|r}
 & 7 & -11 & 0 & -94 & 7 \\
3 & & 21 & 30 & 90 & -12 \\ \hline
 & 7 & 10 & 30 & -4 & \boxed{-5}
\end{array}$$

COEFFICIENTS OF THE QUOTIENT REMAINDER

The steps with numbers in green are the same as the steps in the division above.

This method, where we only use the coefficients and we only complete the really important operations, is called *Ruffini's rule*.

Ruffini's rule can be used to divide a polynomial by $x - a$. Operations such as additions and multiplications are carried out one by one. This gives us the coefficient of the quotient and the rest of the division.

Factoring

If we know how to divide polynomials, we can separate a polynomial into products of factors. This is called **factoring the polynomial.** In this process, Ruffini's rule is very useful.

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Worked examples

1. ***Divide the polynomial.***

$P(x) = 6x^4 + 8x^2 + 7x + 40$

by $Q(x) = 2x^2 - 4x + 5$.

NOTE: In this division we can't apply Ruffini's rule, because the divisor is not the $x - a$ type. It is a 2nd-degree polynomial.

$$\begin{array}{llllll|l}
6x^4 & & +8x^2 & +7x & +40 & & 2x^2-4x+5 \\ \hline
-6x^4 & +12x^3 & -15x^2 & & & & 3x^2+6x+17/2 \\ \hline
 & 12x^3 & -7x^2 & & & & \\
 & -12x^3 & +24x^2 & -30x & & & \\ \hline
 & & 17x^2 & -23x & & & \\
 & & -17x^2 & +34x & -85/2 & & \\ \hline
 & & & 11x & -5/2 & &
\end{array}$$

$(6x^4) : (2x^2) = \boxed{3x^2}$

$(12x^3) : (2x^2) = \boxed{6x}$

$(17x^2) : (2x^2) = \boxed{17/2}$

The quotient is $C(x) = 3x^2 + 6x + \frac{17}{2}$, and the remainder is $R(x) = 11x - \frac{5}{2}$.

2. ***Divide the polynomial.***

$P(x) = x^3 - 13x + 12$

by $Q(x) = x - 1$.

Is $P(x)$ divisible by $Q(x)$?

Line up the coefficients. Remember that the term in x^2, is missing, so we put a 0 in its place.

$$\begin{array}{r|rrr|r}
 & 1 & 0 & -13 & 12 \\
1 & & 1 & 1 & -12 \\ \hline
 & 1 & 1 & -12 & \boxed{0}
\end{array}$$

QUOTIENT: 1 1 –12, which corresponds with: $x^2 + x - 12$ REMAINDER: 0

So, $P(x) = (x - 1)(x^2 + x - 12)$.

Since the remainder is 0, we can say that the polynomial $P(x)$ is divisible by $Q(x)$.

3. ***Using Ruffini's rule, transform the following polynomial into the product of factors:***

$P(x) = x^3 - 4x^2 + x + 6$

Important rule. If the coefficients of a polynomial are whole numbers, the roots of this polynomial are divisors (positive or negative) of its independent term (the term without an x). We will justify this rule next year.

In this polynomial, $x^3 - 4x^2 + x + 6$, the independent term is 6. Its divisors are 1, –1, 2, –2, 3, –3, 6 and –6. Let's start with the simplest ones.

$$\begin{array}{r|rrr|r}
 & 1 & -4 & 1 & 6 \\
1 & & 1 & -3 & -2 \\ \hline
 & 1 & -3 & -2 & \boxed{4}
\end{array}$$

1 is not a root.

$$\begin{array}{r|rrr|r}
 & 1 & -4 & 1 & 6 \\
-1 & & -1 & 5 & -6 \\ \hline
 & 1 & -5 & 6 & \boxed{0}
\end{array}$$

–1 is a root.

So, the division is exact and $P(x) = (x + 1) \cdot (x^2 - 5x + 6)$ is true.

Now, find the divisors of $x^2 - 5x + 6$. If we try with –1, we see that it is no longer a root. If we try with 2, we can see that it is a root.

$$\begin{array}{r|rr|r}
 & 1 & -5 & 6 \\
2 & & 2 & -6 \\ \hline
 & 1 & -3 & \boxed{0}
\end{array}$$

So, $(x^2 - 5x + 6) = (x - 2)(x - 3)$

That means $P(x) = (x + 1)(x^2 - 5x + 6) = (x + 1)(x - 2)(x - 3)$. We factored the initial polynomial.

Do it yourself. Use Ruffini's rule to transform the following polynomial into the product of factors (first, remove x as the common factor):

$P(x) = x^4 - x^3 - 4x^2 + 4x$

Think and practise

1. Find the quotient and the remainder of these divisions:

a) $(x^5 - 7x^4 + 3x^2 - 8) : (x^2 - 3x + 1)$

b) $(6x^4 + 3x^3 - 2x) : (3x^2 + 2)$

c) $(5x^4 + 6x^2 - 11x + 13) : (x - 2)$ using Ruffini's law.

2. Factor the following polynomials:

a) $P(x) = x^3 - 7x - 6$

b) $P(x) = x^4 + 3x^2 - 4x$

c) $P(x) = x^3 - 3x + 2$

d) $P(x) = x^4 - x^2$

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Suggestions

- After studying the sum, the product of monomials and polynomials and some useful applications such as notable products and the extraction of common factors, we will approach quotients of polynomials, which are associated with the division of natural numbers.
- The division of polynomials involves a series of operations: the division of two monomials, the product of a monomial and polynomials, and the subtraction of polynomials.

  This process, which is described in the margin of the Student's Book, should be repeated as many times as necessary. The biggest challenge for students is to approach the division of polynomials in a systematic and organised way.
- It might be useful for students to follow the steps outlined in the Worked example section: leave a space for the missing terms and annotate the divisions to obtain the terms of the quotient, as well as the multiplications by the divider, so that we can then subtract each term from the product.
- The example we have chosen, with a divider type $x - a$, allows us to use the same division to present Ruffini's rule. Ruffini's rule is a very efficient rule (when it can be applied), and helps us to carry out the process of factoring a polynomial.
- Students need to understand the relationship between traditional division and division using Ruffini's rule. This will ensure that they internalise the rule in a logical way, rather than in an automatic way.

  Although this rule can be learned relatively quickly, it is important that students remember when it can be applied: the divider needs to be type $x - a$ and the remainder is always a number.

  The examples presented in the textbook aim to reinforce this particular concept.
- Ruffini's rule and the factorisation of polynomials will be studied in more detail in the following academic year.

Answers to 'Think and practise'

1 a) Quotient: $x^3 - 4x^2 - 13x - 32$; Remainder: $-83x + 24$

b) Quotient: $2x^2 + x - \frac{4}{3}$; Remainder: $-4x + \frac{8}{3}$

c) Quotient: $5x^3 + 10x^2 + 26x + 41$; Remainder: 95

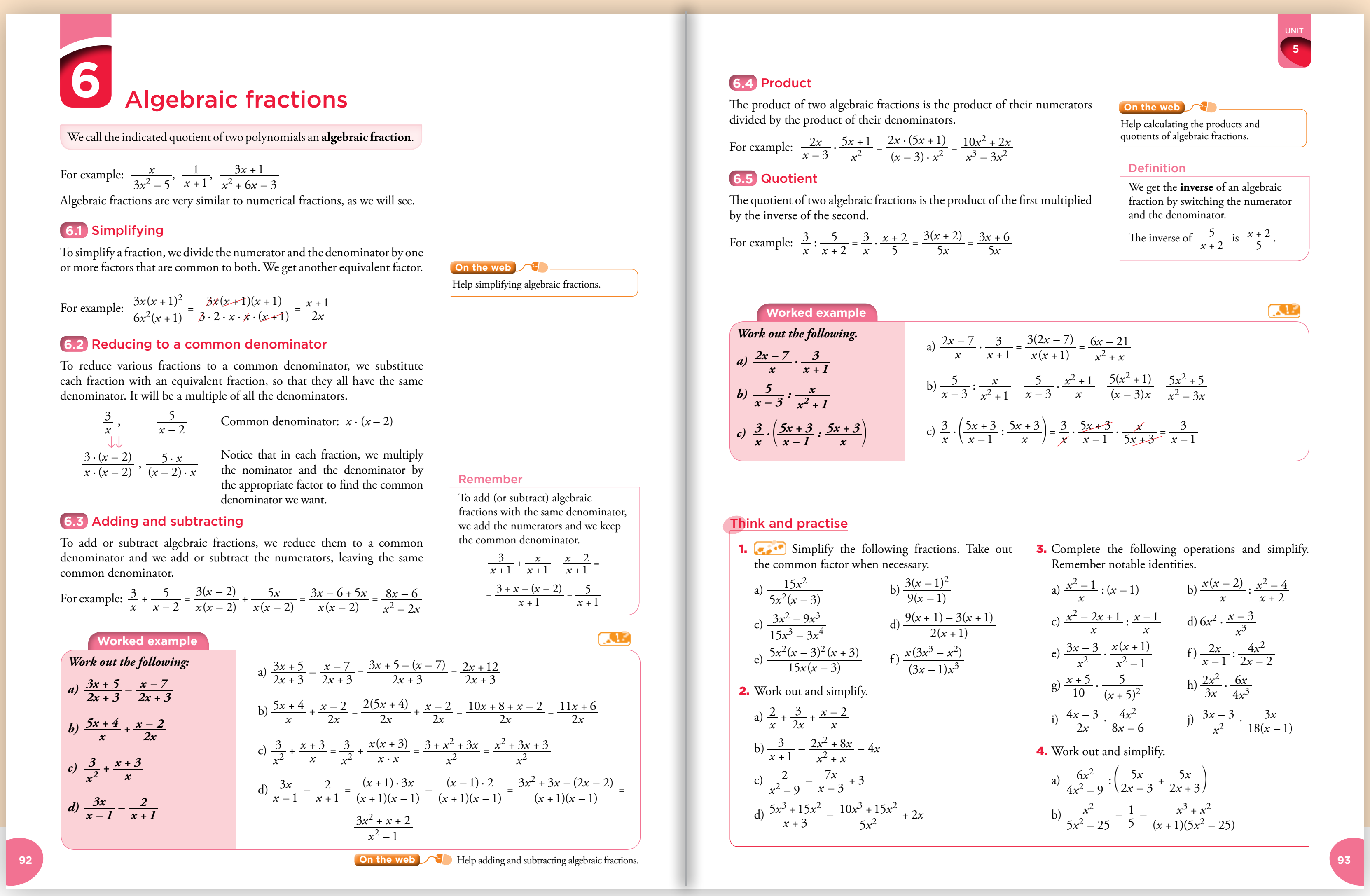
2 a) $(x + 2)(x - 3)(x + 1)$

b) $x(x - 1)(x^2 + x + 4)$

c) $(x + 2)(x - 1)^2$

d) $x^2(x + 1)(x - 1)$

Notes



6 Algebraic fractions

We call the indicated quotient of two polynomials an **algebraic fraction**.

For example: $\frac{x}{3x^2-5}$, $\frac{1}{x+1}$, $\frac{3x+1}{x^2+6x-3}$

Algebraic fractions are very similar to numerical fractions, as we will see.

6.1 Simplifying

To simplify a fraction, we divide the numerator and the denominator by one or more factors that are common to both. We get another equivalent factor.

For example: $\frac{3x(x+1)^2}{6x^2(x+1)} = \frac{\cancel{3}\cancel{x}\cancel{(x+1)}(x+1)}{\cancel{3}\cdot 2\cdot x\cdot \cancel{x}\cdot \cancel{(x+1)}} = \frac{x+1}{2x}$

On the web Help simplifying algebraic fractions.

6.2 Reducing to a common denominator

To reduce various fractions to a common denominator, we substitute each fraction with an equivalent fraction, so that they all have the same denominator. It will be a multiple of all the denominators.

$$\frac{3}{x},\quad \frac{5}{x-2}$$
$$\downarrow\downarrow$$
$$\frac{3\cdot(x-2)}{x\cdot(x-2)},\quad \frac{5\cdot x}{(x-2)\cdot x}$$

Common denominator: $x\cdot(x-2)$

Notice that in each fraction, we multiply the nominator and the denominator by the appropriate factor to find the common denominator we want.

6.3 Adding and subtracting

To add or subtract algebraic fractions, we reduce them to a common denominator and we add or subtract the numerators, leaving the same common denominator.

For example: $\frac{3}{x}+\frac{5}{x-2}=\frac{3(x-2)}{x(x-2)}+\frac{5x}{x(x-2)}=\frac{3x-6+5x}{x(x-2)}=\frac{8x-6}{x^2-2x}$

Remember

To add (or subtract) algebraic fractions with the same denominator, we add the numerators and we keep the common denominator.

$$\frac{3}{x+1}+\frac{x}{x+1}-\frac{x-2}{x+1}=$$
$$=\frac{3+x-(x-2)}{x+1}=\frac{5}{x+1}$$

Worked example

Work out the following:

a) $\frac{3x+5}{2x+3}-\frac{x-7}{2x+3}$

b) $\frac{5x+4}{x}+\frac{x-2}{2x}$

c) $\frac{3}{x^2}+\frac{x+3}{x}$

d) $\frac{3x}{x-1}-\frac{2}{x+1}$

a) $\frac{3x+5}{2x+3}-\frac{x-7}{2x+3}=\frac{3x+5-(x-7)}{2x+3}=\frac{2x+12}{2x+3}$

b) $\frac{5x+4}{x}+\frac{x-2}{2x}=\frac{2(5x+4)}{2x}+\frac{x-2}{2x}=\frac{10x+8+x-2}{2x}=\frac{11x+6}{2x}$

c) $\frac{3}{x^2}+\frac{x+3}{x}=\frac{3}{x^2}+\frac{x(x+3)}{x\cdot x}=\frac{3+x^2+3x}{x^2}=\frac{x^2+3x+3}{x^2}$

d) $\frac{3x}{x-1}-\frac{2}{x+1}=\frac{(x+1)\cdot 3x}{(x+1)(x-1)}-\frac{(x-1)\cdot 2}{(x+1)(x-1)}=\frac{3x^2+3x-(2x-2)}{(x+1)(x-1)}=$

$=\frac{3x^2+x+2}{x^2-1}$

On the web Help adding and subtracting algebraic fractions.

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UNIT 5

6.4 Product

The product of two algebraic fractions is the product of their numerators divided by the product of their denominators.

For example: $\frac{2x}{x-3}\cdot\frac{5x+1}{x^2}=\frac{2x\cdot(5x+1)}{(x-3)\cdot x^2}=\frac{10x^2+2x}{x^3-3x^2}$

6.5 Quotient

The quotient of two algebraic fractions is the product of the first multiplied by the inverse of the second.

For example: $\frac{3}{x}:\frac{5}{x+2}=\frac{3}{x}\cdot\frac{x+2}{5}=\frac{3(x+2)}{5x}=\frac{3x+6}{5x}$

On the web Help calculating the products and quotients of algebraic fractions.

Definition

We get the **inverse** of an algebraic fraction by switching the numerator and the denominator.

The inverse of $\frac{5}{x+2}$ is $\frac{x+2}{5}$.

Worked example

Work out the following.

a) $\frac{2x-7}{x}\cdot\frac{3}{x+1}$

b) $\frac{5}{x-3}:\frac{x}{x^2+1}$

c) $\frac{3}{x}\cdot\left(\frac{5x+3}{x-1}:\frac{5x+3}{x}\right)$

a) $\frac{2x-7}{x}\cdot\frac{3}{x+1}=\frac{3(2x-7)}{x(x+1)}=\frac{6x-21}{x^2+x}$

b) $\frac{5}{x-3}:\frac{x}{x^2+1}=\frac{5}{x-3}\cdot\frac{x^2+1}{x}=\frac{5(x^2+1)}{(x-3)x}=\frac{5x^2+5}{x^2-3x}$

c) $\frac{3}{x}\cdot\left(\frac{5x+3}{x-1}:\frac{5x+3}{x}\right)=\frac{3}{\cancel{x}}\cdot\frac{\cancel{5x+3}}{x-1}\cdot\frac{\cancel{x}}{\cancel{5x+3}}=\frac{3}{x-1}$

Think and practise

1. Simplify the following fractions. Take out the common factor when necessary.

a) $\frac{15x^2}{5x^2(x-3)}$ b) $\frac{3(x-1)^2}{9(x-1)}$

c) $\frac{3x^2-9x^3}{15x^3-3x^4}$ d) $\frac{9(x+1)-3(x+1)}{2(x+1)}$

e) $\frac{5x^2(x-3)^2(x+3)}{15x(x-3)}$ f) $\frac{x(3x^3-x^2)}{(3x-1)x^3}$

2. Work out and simplify.

a) $\frac{2}{x}+\frac{3}{2x}+\frac{x-2}{x}$

b) $\frac{3}{x+1}-\frac{2x^2+8x}{x^2+x}-4x$

c) $\frac{2}{x^2-9}-\frac{7x}{x-3}+3$

d) $\frac{5x^3+15x^2}{x+3}-\frac{10x^3+15x^2}{5x^2}+2x$

3. Complete the following operations and simplify. Remember notable identities.

a) $\frac{x^2-1}{x}:(x-1)$ b) $\frac{x(x-2)}{x}:\frac{x^2-4}{x+2}$

c) $\frac{x^2-2x+1}{x}:\frac{x-1}{x}$ d) $6x^2\cdot\frac{x-3}{x^3}$

e) $\frac{3x-3}{x^2}\cdot\frac{x(x+1)}{x^2-1}$ f) $\frac{2x}{x-1}:\frac{4x^2}{2x-2}$

g) $\frac{x+5}{10}\cdot\frac{5}{(x+5)^2}$ h) $\frac{2x^2}{3x}\cdot\frac{6x}{4x^3}$

i) $\frac{4x-3}{2x}\cdot\frac{4x^2}{8x-6}$ j) $\frac{3x-3}{x^2}\cdot\frac{3x}{18(x-1)}$

4. Work out and simplify.

a) $\frac{6x^2}{4x^2-9}:\left(\frac{5x}{2x-3}+\frac{5x}{2x+3}\right)$

b) $\frac{x^2}{5x^2-25}-\frac{1}{5}-\frac{x^3+x^2}{(x+1)(5x^2-25)}$

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Suggestions

- We begin this section by explaining the meaning and use of one of the most difficult mathematical tools that students will encounter this year.
- It is clear that the challenge resides in the operations required. Even if we explain to students that it is just an extension of the work we have done with fractions, we actually know that simplifying an algebraic fraction involves a range of techniques to transform a polynomial into a product of factors (extracting common factor, recognising identities and applying Ruffini's rule).
- You should begin this process using simple examples in which a common factor can be extracted. Encourage students to simplify common factors in the numerator and denominator. Others, that present identities in the numerator and denominator, can be simplified once they have been expressed as products. Then, use expressions that allow students to use both techniques simultaneously. At this level, we don't consider it appropriate to use fractions that require Ruffini's rule to be simplified.
- In operations involving fractions, avoid excessively complex calculations. For example, the lowest common multiple or the denominators in sums should not be a polynomial with a degree higher than 2; in the product and quotient, insist that multiplications in the numerator and denominator are annotated to see if the fraction can be simplified before giving the answer. Just like with normal fractions, the fractions should be provided in their simplest form.

Answer to 'Think and practise'

1 a) $\frac{3}{x-1}$ b) $\frac{x-1}{3}$

c) $\frac{1-3x}{x(5-x)}=\frac{-3x+1}{-x^2+5x}$ d) 3

e) $\frac{x(x-3)(x+3)}{3}=\frac{x^2-9x}{3}$ f) 1

2 a) $\frac{2x+3}{2x}$ b) $\frac{-4x^2-6x-5}{x+1}$ c) $\frac{-4x^2-21x-25}{x^2-9}$

d) $\frac{25x^5+75x^4-15x^3-45x^2}{5x^3+15x^2}=\frac{5x^3+15x^2-3x-9}{x+3}$

3 a) $\frac{x+1}{x}$ b) 1

c) $x-1$ d) $\frac{6(x-3)}{x}=\frac{6x-18}{x}$

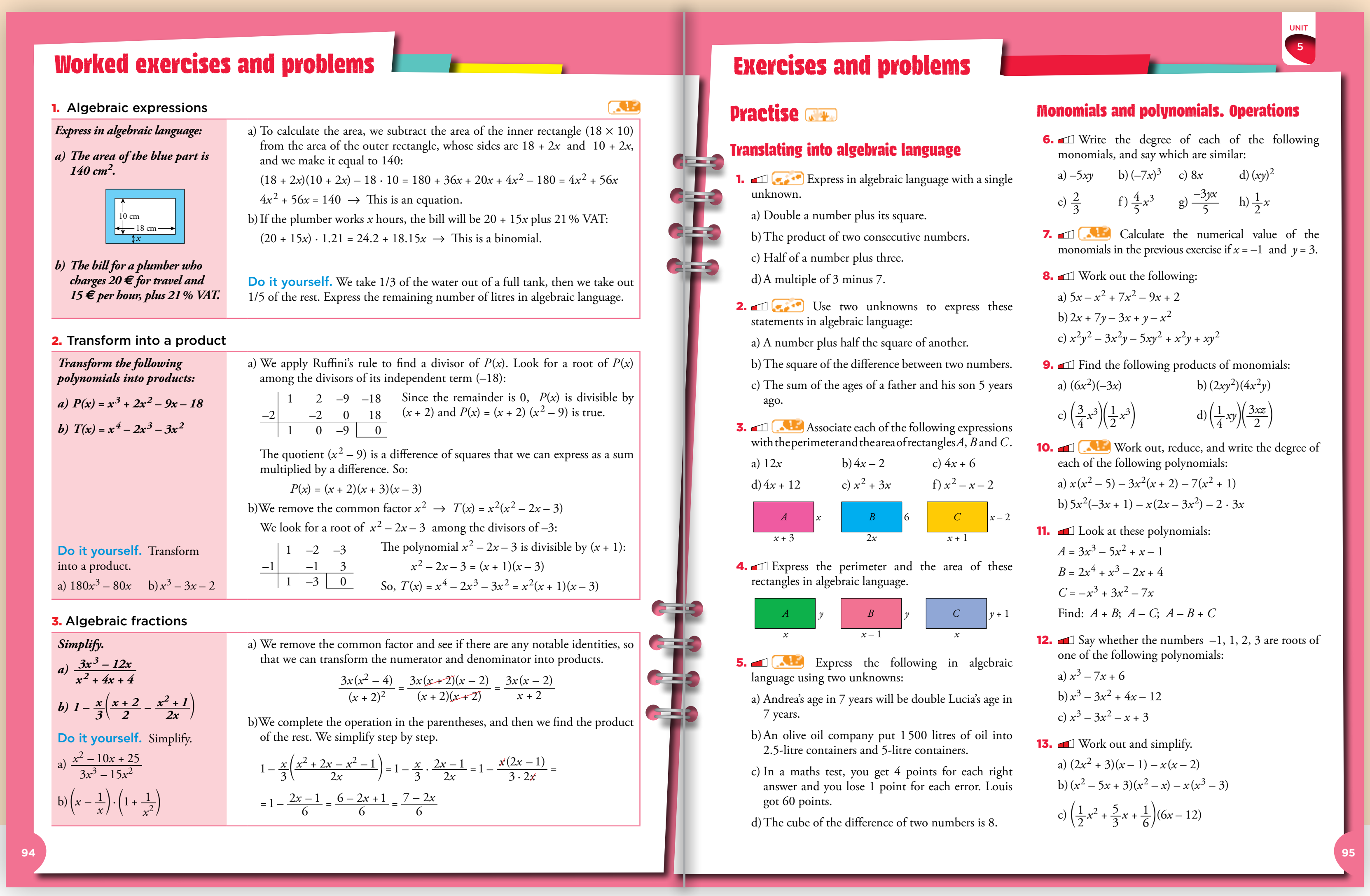
e) $\frac{3}{x}$ f) $\frac{1}{x}$

g) $\frac{1}{2(x+5)}=\frac{1}{2x+10}$ h) $\frac{1}{x}$

i) x j) $\frac{1}{2x}$

4 a) $\frac{3}{10}$ b) $\frac{1}{5}$

Notes



Worked exercises and problems

1. Algebraic expressions

Express in algebraic language:

a) The area of the blue part is 140 cm^2.

b) The bill for a plumber who charges 20 € for travel and 15 € per hour, plus 21 % VAT.

a) To calculate the area, we subtract the area of the inner rectangle (18 × 10) from the area of the outer rectangle, whose sides are $18 + 2x$ and $10 + 2x$, and we make it equal to 140:

$(18 + 2x)(10 + 2x) - 18 \cdot 10 = 180 + 36x + 20x + 4x^2 - 180 = 4x^2 + 56x$

$4x^2 + 56x = 140 \rightarrow$ This is an equation.

b) If the plumber works x hours, the bill will be $20 + 15x$ plus 21 % VAT:

$(20 + 15x) \cdot 1.21 = 24.2 + 18.15x \rightarrow$ This is a binomial.

Do it yourself. We take 1/3 of the water out of a full tank, then we take out 1/5 of the rest. Express the remaining number of litres in algebraic language.

2. Transform into a product

Transform the following polynomials into products:

a) $P(x) = x^3 + 2x^2 - 9x - 18$

b) $T(x) = x^4 - 2x^3 - 3x^2$

Do it yourself. Transform into a product.

a) $180x^3 - 80x$ b) $x^3 - 3x - 2$

a) We apply Ruffini's rule to find a divisor of $P(x)$. Look for a root of $P(x)$ among the divisors of its independent term (–18):

$$\begin{array}{r|rrrr} & 1 & 2 & -9 & -18\\ -2 & & -2 & 0 & 18\\ \hline & 1 & 0 & -9 & \boxed{0}\end{array}$$

Since the remainder is 0, $P(x)$ is divisible by $(x + 2)$ and $P(x) = (x + 2)\,(x^2 - 9)$ is true.

The quotient $(x^2 - 9)$ is a difference of squares that we can express as a sum multiplied by a difference. So:

$P(x) = (x + 2)(x + 3)(x - 3)$

b) We remove the common factor $x^2 \rightarrow T(x) = x^2(x^2 - 2x - 3)$

We look for a root of $x^2 - 2x - 3$ among the divisors of –3:

$$\begin{array}{r|rrr} & 1 & -2 & -3\\ -1 & & -1 & 3\\ \hline & 1 & -3 & \boxed{0}\end{array}$$

The polynomial $x^2 - 2x - 3$ is divisible by $(x + 1)$:

$x^2 - 2x - 3 = (x + 1)(x - 3)$

So, $T(x) = x^4 - 2x^3 - 3x^2 = x^2(x + 1)(x - 3)$

3. Algebraic fractions

Simplify.

a) $\dfrac{3x^3 - 12x}{x^2 + 4x + 4}$

b) $1 - \dfrac{x}{3}\left(\dfrac{x + 2}{2} - \dfrac{x^2 + 1}{2x}\right)$

Do it yourself. Simplify.

a) $\dfrac{x^2 - 10x + 25}{3x^3 - 15x^2}$

b) $\left(x - \dfrac{1}{x}\right) \cdot \left(1 + \dfrac{1}{x^2}\right)$

a) We remove the common factor and see if there are any notable identities, so that we can transform the numerator and denominator into products.

$$\frac{3x(x^2 - 4)}{(x + 2)^2} = \frac{3x\cancel{(x + 2)}(x - 2)}{(x + 2)\cancel{(x + 2)}} = \frac{3x(x - 2)}{x + 2}$$

b) We complete the operation in the parentheses, and then we find the product of the rest. We simplify step by step.

$$1 - \frac{x}{3}\left(\frac{x^2 + 2x - x^2 - 1}{2x}\right) = 1 - \frac{x}{3} \cdot \frac{2x - 1}{2x} = 1 - \frac{\cancel{x}(2x - 1)}{3 \cdot 2\cancel{x}} =$$

$$= 1 - \frac{2x - 1}{6} = \frac{6 - 2x + 1}{6} = \frac{7 - 2x}{6}$$

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UNIT 5

Exercises and problems

Practise

Translating into algebraic language

1. Express in algebraic language with a single unknown.

a) Double a number plus its square.

b) The product of two consecutive numbers.

c) Half of a number plus three.

d) A multiple of 3 minus 7.

2. Use two unknowns to express these statements in algebraic language:

a) A number plus half the square of another.

b) The square of the difference between two numbers.

c) The sum of the ages of a father and his son 5 years ago.

3. Associate each of the following expressions with the perimeter and the area of rectangles A, B and C.

a) $12x$ b) $4x - 2$ c) $4x + 6$

d) $4x + 12$ e) $x^2 + 3x$ f) $x^2 - x - 2$

4. Express the perimeter and the area of these rectangles in algebraic language.

5. Express the following in algebraic language using two unknowns:

a) Andrea's age in 7 years will be double Lucia's age in 7 years.

b) An olive oil company put 1 500 litres of oil into 2.5-litre containers and 5-litre containers.

c) In a maths test, you get 4 points for each right answer and you lose 1 point for each error. Louis got 60 points.

d) The cube of the difference of two numbers is 8.

Monomials and polynomials. Operations

6. Write the degree of each of the following monomials, and say which are similar:

a) $-5xy$ b) $(-7x)^3$ c) $8x$ d) $(xy)^2$

e) $\dfrac{2}{3}$ f) $\dfrac{4}{5}x^3$ g) $\dfrac{-3yx}{5}$ h) $\dfrac{1}{2}x$

7. Calculate the numerical value of the monomials in the previous exercise if $x = -1$ and $y = 3$.

8. Work out the following:

a) $5x - x^2 + 7x^2 - 9x + 2$

b) $2x + 7y - 3x + y - x^2$

c) $x^2y^2 - 3x^2y - 5xy^2 + x^2y + xy^2$

9. Find the following products of monomials:

a) $(6x^2)(-3x)$ b) $(2xy^2)(4x^2y)$

c) $\left(\dfrac{3}{4}x^3\right)\left(\dfrac{1}{2}x^3\right)$ d) $\left(\dfrac{1}{4}xy\right)\left(\dfrac{3xz}{2}\right)$

10. Work out, reduce, and write the degree of each of the following polynomials:

a) $x(x^2 - 5) - 3x^2(x + 2) - 7(x^2 + 1)$

b) $5x^2(-3x + 1) - x(2x - 3x^2) - 2 \cdot 3x$

11. Look at these polynomials:

$A = 3x^3 - 5x^2 + x - 1$

$B = 2x^4 + x^3 - 2x + 4$

$C = -x^3 + 3x^2 - 7x$

Find: $A + B$; $A - C$; $A - B + C$

12. Say whether the numbers –1, 1, 2, 3 are roots of one of the following polynomials:

a) $x^3 - 7x + 6$

b) $x^3 - 3x^2 + 4x - 12$

c) $x^3 - 3x^2 - x + 3$

13. Work out and simplify.

a) $(2x^2 + 3)(x - 1) - x(x - 2)$

b) $(x^2 - 5x + 3)(x^2 - x) - x(x^3 - 3)$

c) $\left(\dfrac{1}{2}x^2 + \dfrac{5}{3}x + \dfrac{1}{6}\right)(6x - 12)$

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Suggestions

- The 'Worked exercises and problems' display strategies, suggestions and hints that students will find useful when solving the activities in the final pages of the unit.
- The final aim of this section is for students to be able to reproduce similar procedures when solving other mathematical problems.

Answers to 'Do it yourself'

1 $\dfrac{8x}{15}$

2 a) $20x(3x + 2)(3x - 2)$ b) $(x - 2)(x + 1)^2$

3 a) $\dfrac{x - 5}{3x^2}$ b) $\dfrac{x^4 - 1}{x^3}$

Answers to 'Exercises and problems'

1 a) $2x + x^2$ b) $x(x + 1)$ c) $\dfrac{(x + 3)}{2}$ d) $3x - 7$

2 a) $x + \dfrac{y^2}{2}$ b) $(x - y)^2$ c) $(x - 5) + (y - 5)$

3 a) $12x$ is the area of B. b) $4x - 2$ is the perimeter of C.

c) $4x + 6$ is the perimeter of A. d) $4x + 12$ is the perimeter of B.

e) $x^2 + 3x$ is the area of A. f) $x^2 - x - 2$ is the area of C.

4 $A\begin{cases}\text{Perimeter} = 2(x + y) = 2x + 2y\\ \text{Area} = xy\end{cases}$

$B\begin{cases}\text{Perimeter} = 2(x - 1 + y) = 2x + 2y - 2\\ \text{Area} = (x - 1)\,y = xy - y\end{cases}$

$C\begin{cases}\text{Perimeter} = 2(x + y + 1) = 2x + 2y + 2\\ \text{Area} = x(y + 1) = xy + x\end{cases}$

5 a) $x + 7 = 2y$ b) $2.5x + 5y = 1\,500$

c) $4x - y = 60$ d) $(x - y)^3 = 8$

6 a) 2 b) 3 c) 1 d) 4 e) 0 f) 3 g) 2 h) 1

Similar: a) and g); b) and f); c) and h)

7 a) 15 b) 343 c) –8 d) 9

e) 2/3 f) –4/5 g) 9/5 h) –1/2

8 a) $6x^2 - 4x + 2$ b) $-x^2 - x + 8y$

c) $x^2y^2 - 2x^2y - 4xy^2$

9 a) $-18x^3$ b) $8x^3y^3$ c) x^6 d) x^2yz

10 a) $-2x^3 - 13x^2 - 5x - 7 \rightarrow$ degree 3

b) $-12x^3 + 3x^2 - 6x \rightarrow$ degree 3

11 $A + B = 2x^4 + 4x^3 - 5x^2 - x + 3$

$A - C = 4x^3 - 8x^2 + 8x - 1$

$A - B + C = -2x^4 + x^3 - 2x^2 - 4x - 5$

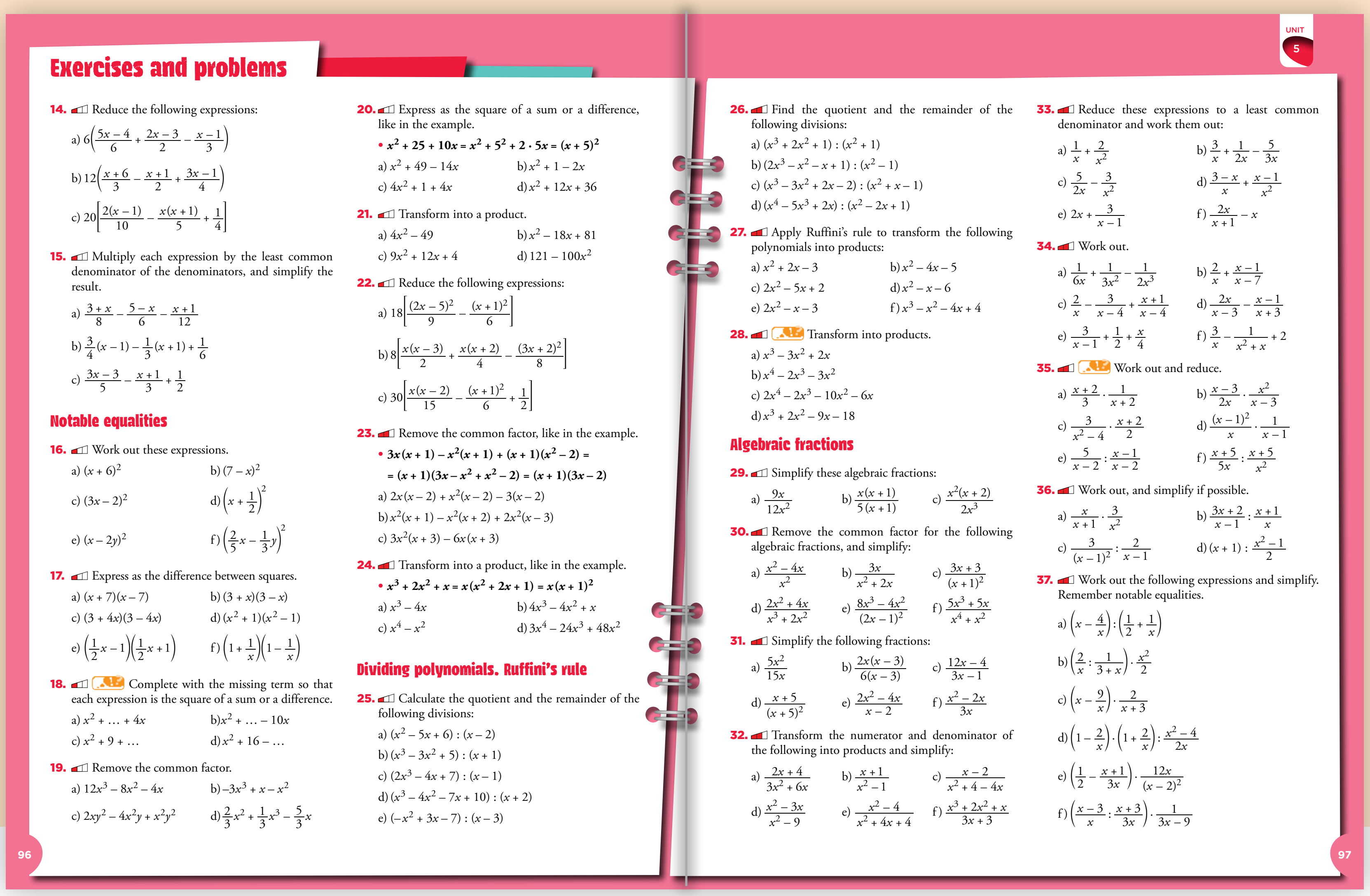
12 a) 1 and 2 are the roots of $x^3 - 7x + 6$.

b) 3 is the root of $x^3 - 3x^2 + 4x - 12$.

c) –1, 1 and 3 are the roots of $x^3 - 3x^2 - x + 3$.

13 a) $2x^3 - 3x^2 + 5x - 3$ b) $-6x^3 + 8x^2$

c) $3x^3 + 4x^2 - 19x - 2$



Exercises and problems

14. Reduce the following expressions:

a) $6\left(\frac{5x-4}{6}+\frac{2x-3}{2}-\frac{x-1}{3}\right)$

b) $12\left(\frac{x+6}{3}-\frac{x+1}{2}+\frac{3x-1}{4}\right)$

c) $20\left[\frac{2(x-1)}{10}-\frac{x(x+1)}{5}+\frac{1}{4}\right]$

15. Multiply each expression by the least common denominator of the denominators, and simplify the result.

a) $\frac{3+x}{8}-\frac{5-x}{6}-\frac{x+1}{12}$

b) $\frac{3}{4}(x-1)-\frac{1}{3}(x+1)+\frac{1}{6}$

c) $\frac{3x-3}{5}-\frac{x+1}{3}+\frac{1}{2}$

Notable equalities

16. Work out these expressions.

a) $(x+6)^2$ b) $(7-x)^2$

c) $(3x-2)^2$ d) $\left(x+\frac{1}{2}\right)^2$

e) $(x-2y)^2$ f) $\left(\frac{2}{5}x-\frac{1}{3}y\right)^2$

17. Express as the difference between squares.

a) $(x+7)(x-7)$ b) $(3+x)(3-x)$

c) $(3+4x)(3-4x)$ d) $(x^2+1)(x^2-1)$

e) $\left(\frac{1}{2}x-1\right)\left(\frac{1}{2}x+1\right)$ f) $\left(1+\frac{1}{x}\right)\left(1-\frac{1}{x}\right)$

18. Complete with the missing term so that each expression is the square of a sum or a difference.

a) $x^2+\ldots+4x$ b) $x^2+\ldots-10x$

c) $x^2+9+\ldots$ d) $x^2+16-\ldots$

19. Remove the common factor.

a) $12x^3-8x^2-4x$ b) $-3x^3+x-x^2$

c) $2xy^2-4x^2y+x^2y^2$ d) $\frac{2}{3}x^2+\frac{1}{3}x^3-\frac{5}{3}x$

20. Express as the square of a sum or a difference, like in the example.

- **$x^2+25+10x=x^2+5^2+2\cdot 5x=(x+5)^2$**

a) $x^2+49-14x$ b) x^2+1-2x

c) $4x^2+1+4x$ d) $x^2+12x+36$

21. Transform into a product.

a) $4x^2-49$ b) $x^2-18x+81$

c) $9x^2+12x+4$ d) $121-100x^2$

22. Reduce the following expressions:

a) $18\left[\frac{(2x-5)^2}{9}-\frac{(x+1)^2}{6}\right]$

b) $8\left[\frac{x(x-3)}{2}+\frac{x(x+2)}{4}-\frac{(3x+2)^2}{8}\right]$

c) $30\left[\frac{x(x-2)}{15}-\frac{(x+1)^2}{6}+\frac{1}{2}\right]$

23. Remove the common factor, like in the example.

- **$3x(x+1)-x^2(x+1)+(x+1)(x^2-2)=$**
$=(x+1)(3x-x^2+x^2-2)=(x+1)(3x-2)$

a) $2x(x-2)+x^2(x-2)-3(x-2)$

b) $x^2(x+1)-x^2(x+2)+2x^2(x-3)$

c) $3x^2(x+3)-6x(x+3)$

24. Transform into a product, like in the example.

- **$x^3+2x^2+x=x(x^2+2x+1)=x(x+1)^2$**

a) x^3-4x b) $4x^3-4x^2+x$

c) x^4-x^2 d) $3x^4-24x^3+48x^2$

Dividing polynomials. Ruffini's rule

25. Calculate the quotient and the remainder of the following divisions:

a) $(x^2-5x+6):(x-2)$

b) $(x^3-3x^2+5):(x+1)$

c) $(2x^3-4x+7):(x-1)$

d) $(x^3-4x^2-7x+10):(x+2)$

e) $(-x^2+3x-7):(x-3)$

26. Find the quotient and the remainder of the following divisions:

a) $(x^3+2x^2+1):(x^2+1)$

b) $(2x^3-x^2-x+1):(x^2-1)$

c) $(x^3-3x^2+2x-2):(x^2+x-1)$

d) $(x^4-5x^3+2x):(x^2-2x+1)$

27. Apply Ruffini's rule to transform the following polynomials into products:

a) x^2+2x-3 b) x^2-4x-5

c) $2x^2-5x+2$ d) x^2-x-6

e) $2x^2-x-3$ f) x^3-x^2-4x+4

28. Transform into products.

a) x^3-3x^2+2x

b) $x^4-2x^3-3x^2$

c) $2x^4-2x^3-10x^2-6x$

d) $x^3+2x^2-9x-18$

Algebraic fractions

29. Simplify these algebraic fractions:

a) $\frac{9x}{12x^2}$ b) $\frac{x(x+1)}{5(x+1)}$ c) $\frac{x^2(x+2)}{2x^3}$

30. Remove the common factor for the following algebraic fractions, and simplify:

a) $\frac{x^2-4x}{x^2}$ b) $\frac{3x}{x^2+2x}$ c) $\frac{3x+3}{(x+1)^2}$

d) $\frac{2x^2+4x}{x^3+2x^2}$ e) $\frac{8x^3-4x^2}{(2x-1)^2}$ f) $\frac{5x^3+5x}{x^4+x^2}$

31. Simplify the following fractions:

a) $\frac{5x^2}{15x}$ b) $\frac{2x(x-3)}{6(x-3)}$ c) $\frac{12x-4}{3x-1}$

d) $\frac{x+5}{(x+5)^2}$ e) $\frac{2x^2-4x}{x-2}$ f) $\frac{x^2-2x}{3x}$

32. Transform the numerator and denominator of the following into products and simplify:

a) $\frac{2x+4}{3x^2+6x}$ b) $\frac{x+1}{x^2-1}$ c) $\frac{x-2}{x^2+4-4x}$

d) $\frac{x^2-3x}{x^2-9}$ e) $\frac{x^2-4}{x^2+4x+4}$ f) $\frac{x^3+2x^2+x}{3x+3}$

33. Reduce these expressions to a least common denominator and work them out:

a) $\frac{1}{x}+\frac{2}{x^2}$ b) $\frac{3}{x}+\frac{1}{2x}-\frac{5}{3x}$

c) $\frac{5}{2x}-\frac{3}{x^2}$ d) $\frac{3-x}{x}+\frac{x-1}{x^2}$

e) $2x+\frac{3}{x-1}$ f) $\frac{2x}{x+1}-x$

34. Work out.

a) $\frac{1}{6x}+\frac{1}{3x^2}-\frac{1}{2x^3}$ b) $\frac{2}{x}+\frac{x-1}{x-7}$

c) $\frac{2}{x}-\frac{3}{x-4}+\frac{x+1}{x-4}$ d) $\frac{2x}{x-3}-\frac{x-1}{x+3}$

e) $\frac{3}{x-1}+\frac{1}{2}+\frac{x}{4}$ f) $\frac{3}{x}-\frac{1}{x^2+x}+2$

35. Work out and reduce.

a) $\frac{x+2}{3}\cdot\frac{1}{x+2}$ b) $\frac{x-3}{2x}\cdot\frac{x^2}{x-3}$

c) $\frac{3}{x^2-4}\cdot\frac{x+2}{2}$ d) $\frac{(x-1)^2}{x}\cdot\frac{1}{x-1}$

e) $\frac{5}{x-2}:\frac{x-1}{x-2}$ f) $\frac{x+5}{5x}:\frac{x+5}{x^2}$

36. Work out, and simplify if possible.

a) $\frac{x}{x+1}\cdot\frac{3}{x^2}$ b) $\frac{3x+2}{x-1}:\frac{x+1}{x}$

c) $\frac{3}{(x-1)^2}:\frac{2}{x-1}$ d) $(x+1):\frac{x^2-1}{2}$

37. Work out the following expressions and simplify. Remember notable equalities.

a) $\left(x-\frac{4}{x}\right):\left(\frac{1}{2}+\frac{1}{x}\right)$

b) $\left(\frac{2}{x}:\frac{1}{3+x}\right)\cdot\frac{x^2}{2}$

c) $\left(x-\frac{9}{x}\right)\cdot\frac{2}{x+3}$

d) $\left(1-\frac{2}{x}\right)\cdot\left(1+\frac{2}{x}\right):\frac{x^2-4}{2x}$

e) $\left(\frac{1}{2}-\frac{x+1}{3x}\right)\cdot\frac{12x}{(x-2)^2}$

f) $\left(\frac{x-3}{x}:\frac{x+3}{3x}\right)\cdot\frac{1}{3x-9}$

Answers to 'Exercises and problems'

14 a) $9x-12$ b) $7x+15$ c) $-3x^2-14x+10$

15 a) $5x-13$ b) $5x-11$ c) $8x-13$

16 a) $x^2+36+12x$ b) $49+x^2-14x$

c) $9x^2+4-12x$ d) $x^2+\frac{1}{4}+x$

e) x^2+4y^2-4xy f) $\frac{4}{25}x^2+\frac{1}{9}y^2-\frac{4}{15}xy$

17 a) x^2-49 b) $9-x^2$ c) $9-16x^2$

d) x^4-1 e) $\frac{1}{4}x^2-1$ f) $1-\frac{1}{x^2}$

18 a) x^2+4+4x b) $x^2+25-10x$

c) x^2+9+6x d) $x^2+16+8x$

19 a) $4x(3x^2-2x-1)$ b) $(-3x^2+1-x)$

c) $xy(2y-4x+xy)$ d) $\frac{1}{3}x(2x+x^2-5)$

20 a) $(x-7)^2$ b) $(x-1)^2$

c) $(2x+1)^2$ d) $(x+6)^2$

21 a) $(2x-7)(2x+7)$ b) $(x-9)^2$

c) $(3x+2)^2$ d) $(11+10x)(11-10x)$

22 a) $5x^2-46x+47$ b) $-3x^2-20x-4$

c) $-3x^2-14x+10$

23 a) $(x-2)(2x+x^2-3)$ b) $x^2(2x-7)$

c) $3x(x-2)(x+3)$

24 a) $x(x+2)(x-2)$ b) $x(2x-1)^2$

c) $x^2(x+1)(x-1)$ d) $3x^2(x-4)^2$

25 a) Quotient: $x-3$; Remainder: 0

b) Quotient: x^2-4x+4; Remainder: 1

c) Quotient: $2x^2+2x-2$; Remainder: 5

d) Quotient: x^2-6x+5; Remainder: 0

e) Quotient: $-x$; Remainder: -7

26 a) Quotient: $x+2$; Remainder: $-x-1$

b) Quotient: $2x-1$; Remainder: x

c) Quotient: $x-4$; Remainder: $7x-6$

d) Quotient: x^2-3x-7; Remainder: $-9x+7$

27 a) $(x+3)(x-1)$ b) $(x-5)(x+1)$ c) $2(x-2)\left(x-\frac{1}{2}\right)$

d) $(x-3)(x+2)$ e) $2(x+1)\left(x-\frac{3}{2}\right)$ f) $(x-1)(x-2)(x+2)$

28 a) $x(x-1)(x-2)$ b) $x^2(x-3)(x+1)$

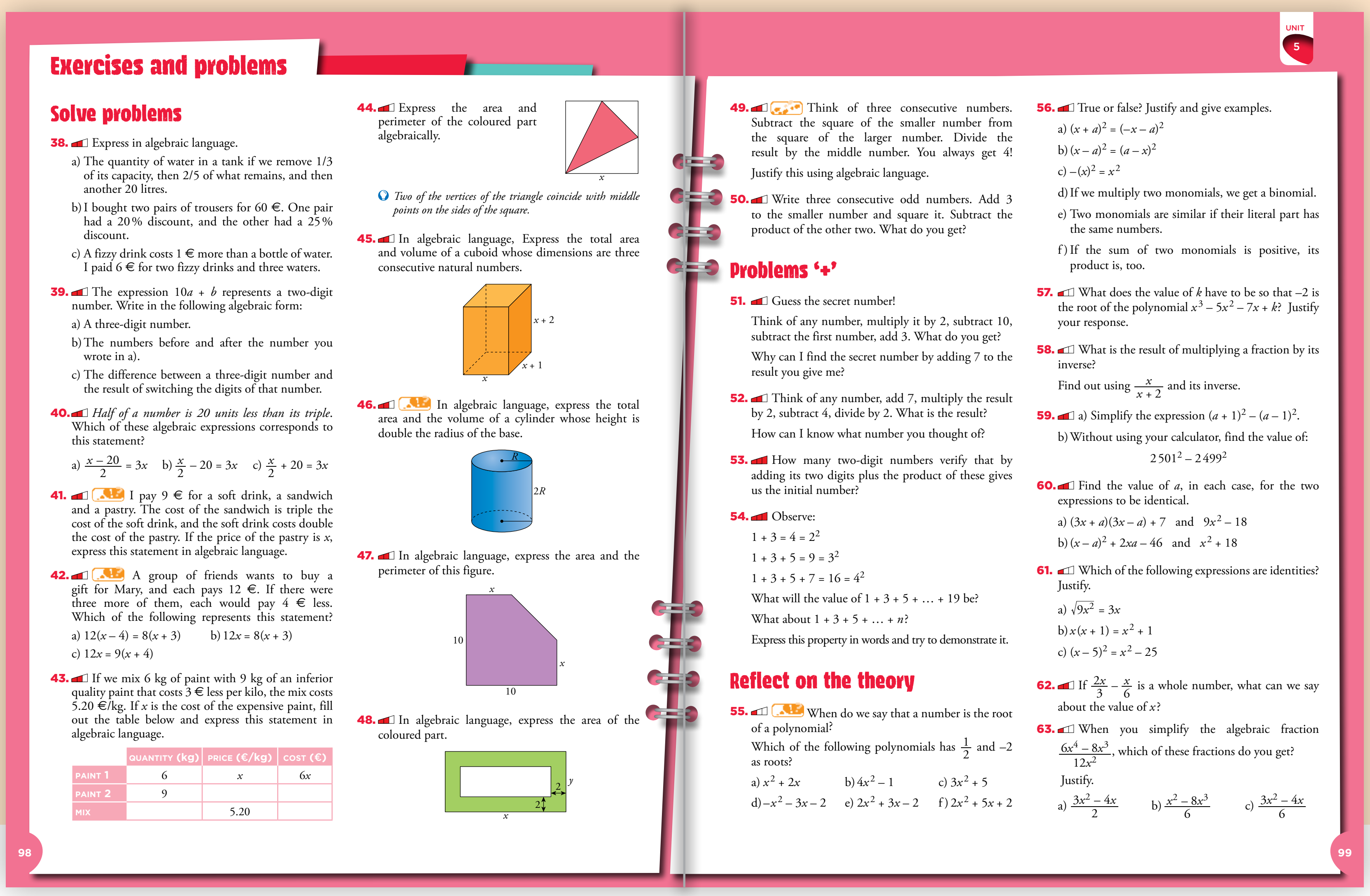
c) $2x(x+1)(x+1)(x-3)$ d) $(x-3)(x+2)(x+3)$

29 a) $\frac{3}{4x}$ b) $\frac{x}{5}$ c) $\frac{x+2}{2x}$

30 a) $\frac{x-4}{x}$ b) $\frac{3}{x+2}$ c) $\frac{3}{x+1}$

d) $\frac{2}{x}$ e) $\frac{4x^2}{2x-1}$ f) $\frac{5}{x}$

31 to 37 Answers at the end of the unit.



Exercises and problems

Solve problems

38. Express in algebraic language.

a) The quantity of water in a tank if we remove 1/3 of its capacity, then 2/5 of what remains, and then another 20 litres.

b) I bought two pairs of trousers for 60 €. One pair had a 20 % discount, and the other had a 25 % discount.

c) A fizzy drink costs 1 € more than a bottle of water. I paid 6 € for two fizzy drinks and three waters.

39. The expression $10a + b$ represents a two-digit number. Write in the following algebraic form:

a) A three-digit number.

b) The numbers before and after the number you wrote in a).

c) The difference between a three-digit number and the result of switching the digits of that number.

40. *Half of a number is 20 units less than its triple.* Which of these algebraic expressions corresponds to this statement?

a) $\frac{x-20}{2} = 3x$ b) $\frac{x}{2} - 20 = 3x$ c) $\frac{x}{2} + 20 = 3x$

41. I pay 9 € for a soft drink, a sandwich and a pastry. The cost of the sandwich is triple the cost of the soft drink, and the soft drink costs double the cost of the pastry. If the price of the pastry is x, express this statement in algebraic language.

42. A group of friends wants to buy a gift for Mary, and each pays 12 €. If there were three more of them, each would pay 4 € less. Which of the following represents this statement?

a) $12(x - 4) = 8(x + 3)$ b) $12x = 8(x + 3)$

c) $12x = 9(x + 4)$

43. If we mix 6 kg of paint with 9 kg of an inferior quality paint that costs 3 € less per kilo, the mix costs 5.20 €/kg. If x is the cost of the expensive paint, fill out the table below and express this statement in algebraic language.

	QUANTITY (kg)	PRICE (€/kg)	COST (€)
PAINT 1	6	x	$6x$
PAINT 2	9		
MIX		5.20	

44. Express the area and perimeter of the coloured part algebraically.

Two of the vertices of the triangle coincide with middle points on the sides of the square.

45. In algebraic language, Express the total area and volume of a cuboid whose dimensions are three consecutive natural numbers.

46. In algebraic language, express the total area and the volume of a cylinder whose height is double the radius of the base.

47. In algebraic language, express the area and the perimeter of this figure.

48. In algebraic language, express the area of the coloured part.

49. Think of three consecutive numbers. Subtract the square of the smaller number from the square of the larger number. Divide the result by the middle number. You always get 4!

Justify this using algebraic language.

50. Write three consecutive odd numbers. Add 3 to the smaller number and square it. Subtract the product of the other two. What do you get?

Problems '+'

51. Guess the secret number!

Think of any number, multiply it by 2, subtract 10, subtract the first number, add 3. What do you get?

Why can I find the secret number by adding 7 to the result you give me?

52. Think of any number, add 7, multiply the result by 2, subtract 4, divide by 2. What is the result?

How can I know what number you thought of?

53. How many two-digit numbers verify that by adding its two digits plus the product of these gives us the initial number?

54. Observe:

$1 + 3 = 4 = 2^2$

$1 + 3 + 5 = 9 = 3^2$

$1 + 3 + 5 + 7 = 16 = 4^2$

What will the value of $1 + 3 + 5 + \ldots + 19$ be?

What about $1 + 3 + 5 + \ldots + n$?

Express this property in words and try to demonstrate it.

Reflect on the theory

55. When do we say that a number is the root of a polynomial?

Which of the following polynomials has $\frac{1}{2}$ and –2 as roots?

a) $x^2 + 2x$ b) $4x^2 - 1$ c) $3x^2 + 5$

d) $-x^2 - 3x - 2$ e) $2x^2 + 3x - 2$ f) $2x^2 + 5x + 2$

56. True or false? Justify and give examples.

a) $(x + a)^2 = (-x - a)^2$

b) $(x - a)^2 = (a - x)^2$

c) $-(x)^2 = x^2$

d) If we multiply two monomials, we get a binomial.

e) Two monomials are similar if their literal part has the same numbers.

f) If the sum of two monomials is positive, its product is, too.

57. What does the value of k have to be so that –2 is the root of the polynomial $x^3 - 5x^2 - 7x + k$? Justify your response.

58. What is the result of multiplying a fraction by its inverse?

Find out using $\frac{x}{x+2}$ and its inverse.

59. a) Simplify the expression $(a + 1)^2 - (a - 1)^2$.

b) Without using your calculator, find the value of:

$$2\,501^2 - 2\,499^2$$

60. Find the value of a, in each case, for the two expressions to be identical.

a) $(3x + a)(3x - a) + 7$ and $9x^2 - 18$

b) $(x - a)^2 + 2xa - 46$ and $x^2 + 18$

61. Which of the following expressions are identities? Justify.

a) $\sqrt{9x^2} = 3x$

b) $x(x + 1) = x^2 + 1$

c) $(x - 5)^2 = x^2 - 25$

62. If $\frac{2x}{3} - \frac{x}{6}$ is a whole number, what can we say about the value of x?

63. When you simplify the algebraic fraction $\frac{6x^4 - 8x^3}{12x^2}$, which of these fractions do you get?

Justify.

a) $\frac{3x^2 - 4x}{2}$ b) $\frac{x^2 - 8x^3}{6}$ c) $\frac{3x^2 - 4x}{6}$

Answers to 'Exercises and problems'

38 a) $x - \frac{x}{3} - \frac{2}{5}\left(x - \frac{x}{3}\right) - 20 = 0$

b) $0.8x + 0.75y = 60$

c) If x is the value of the water bottle, $3(x + 1) + 2x = 6$

If x is the price of the fizzy drink, $3x + 2(x - 1) = 6$

39 a) $100a + 10b + c$

b) $100a + 10b + c + 1$ and $100a + 10b + c - 1$

c) $(100a + 10b + c) - (100c + 10b + a) = 99a - 99c$

40 It is c).

41 $x + 2x + 6x = 9$

42 b).

43

	QUANTITY (kg)	PRICE (€/kg)	COST (€)
PAINT 1	6	x	$6x$
PAINT 2	9	$x - 3$	$9(x - 3)$
MIX	15	5.20	$6x + 9(x - 3)$

Cost of mix → $\frac{6x + 9(x - 3)}{15} = 5.20$

44 Perimeter = $\frac{2\sqrt{5} + \sqrt{2}}{2}x$ Area = $\frac{3}{8}x^2$

45 Area = $6x^2 + 12x + 4$ Volume = $x^3 + 3x^2 + 2x$

46 Area = $6\pi R^2$ Volume = $2\pi R^3$

47 Perimeter = $20 + 2x + 5\sqrt{2}$ Area = $20x - \frac{25}{2}$

48 $A = 4x + 4y - 16$

49 $(x^2 + 2)^2 - x^2 = 4x + 4 \rightarrow \frac{4x + 4}{x + 1} = 4$

50 You always get 1.

51 x; $2x$; $2x - 10$; $2x - 10 - x = x - 10$; $x - 10 + 3 = x - 7$

52 By subtracting five from the result.

53 All of the two-digit numbers with a nine at the end.

54 $1 + 3 + \ldots + 19 = 10^2$

$1 + 3 + \ldots + n = n^2$

55 A number, a, is the root of a polynomial, $P(x)$, if $P(a) = 0$.

Polynomial e).

56 a) T b) T c) F d) F e) F f) F

57 $k = 4$

58 The result is 1.

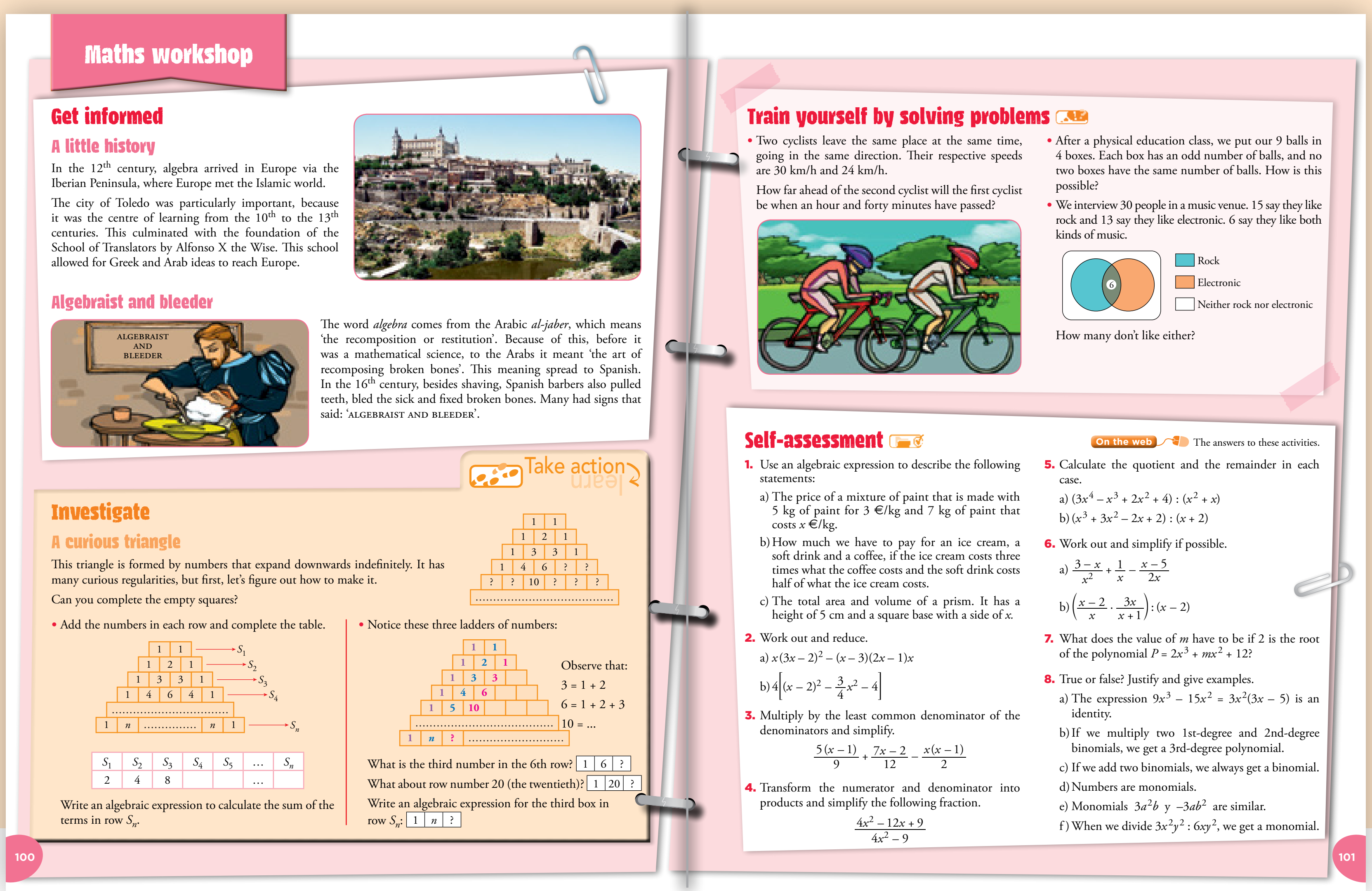
59 a) $4a$ b) 10 000

60 a) $a = 5$ or $a = -5$ b) $a = 8$ or $a = -8$

61 a) is the only identity.

62 x is an even number.

63 You get fraction c).



Maths workshop

Get informed

A little history

In the 12th century, algebra arrived in Europe via the Iberian Peninsula, where Europe met the Islamic world.

The city of Toledo was particularly important, because it was the centre of learning from the 10th to the 13th centuries. This culminated with the foundation of the School of Translators by Alfonso X the Wise. This school allowed for Greek and Arab ideas to reach Europe.

Algebraist and bleeder

The word *algebra* comes from the Arabic *al-jaber*, which means 'the recomposition or restitution'. Because of this, before it was a mathematical science, to the Arabs it meant 'the art of recomposing broken bones'. This meaning spread to Spanish. In the 16th century, besides shaving, Spanish barbers also pulled teeth, bled the sick and fixed broken bones. Many had signs that said: 'ALGEBRAIST AND BLEEDER'.

Take action
learn

Investigate

A curious triangle

This triangle is formed by numbers that expand downwards indefinitely. It has many curious regularities, but first, let's figure out how to make it.

Can you complete the empty squares?

1 1
1 2 1
1 3 3 1
1 4 6 ? ?
? ? 10 ? ? ?
....

- Add the numbers in each row and complete the table.

1 1 → S_1
1 2 1 → S_2
1 3 3 1 → S_3
1 4 6 4 1 → S_4
....
1 n .... n 1 → S_n

S_1	S_2	S_3	S_4	S_5	…	S_n
2	4	8			…	

Write an algebraic expression to calculate the sum of the terms in row S_n.

- Notice these three ladders of numbers:

1 1
1 2 1
1 3 3
1 4 6
1 5 10
....
1 n ? ....

Observe that:
3 = 1 + 2
6 = 1 + 2 + 3
10 = ...

What is the third number in the 6th row? | 1 | 6 | ? |

What about row number 20 (the twentieth)? | 1 | 20 | ? |

Write an algebraic expression for the third box in row S_n: | 1 | n | ? |

100

Train yourself by solving problems

- Two cyclists leave the same place at the same time, going in the same direction. Their respective speeds are 30 km/h and 24 km/h.

  How far ahead of the second cyclist will the first cyclist be when an hour and forty minutes have passed?

- After a physical education class, we put our 9 balls in 4 boxes. Each box has an odd number of balls, and no two boxes have the same number of balls. How is this possible?
- We interview 30 people in a music venue. 15 say they like rock and 13 say they like electronic. 6 say they like both kinds of music.

🟦 Rock
🟧 Electronic
☐ Neither rock nor electronic

How many don't like either?

Self-assessment

On the web The answers to these activities.

1. Use an algebraic expression to describe the following statements:

a) The price of a mixture of paint that is made with 5 kg of paint for 3 €/kg and 7 kg of paint that costs x €/kg.

b) How much we have to pay for an ice cream, a soft drink and a coffee, if the ice cream costs three times what the coffee costs and the soft drink costs half of what the ice cream costs.

c) The total area and volume of a prism. It has a height of 5 cm and a square base with a side of x.

2. Work out and reduce.

a) $x(3x-2)^2-(x-3)(2x-1)x$

b) $4\left[(x-2)^2-\frac{3}{4}x^2-4\right]$

3. Multiply by the least common denominator of the denominators and simplify.

$$\frac{5(x-1)}{9}+\frac{7x-2}{12}-\frac{x(x-1)}{2}$$

4. Transform the numerator and denominator into products and simplify the following fraction.

$$\frac{4x^2-12x+9}{4x^2-9}$$

5. Calculate the quotient and the remainder in each case.

a) $(3x^4-x^3+2x^2+4):(x^2+x)$

b) $(x^3+3x^2-2x+2):(x+2)$

6. Work out and simplify if possible.

a) $\frac{3-x}{x^2}+\frac{1}{x}-\frac{x-5}{2x}$

b) $\left(\frac{x-2}{x}\cdot\frac{3x}{x+1}\right):(x-2)$

7. What does the value of m have to be if 2 is the root of the polynomial $P=2x^3+mx^2+12$?

8. True or false? Justify and give examples.

a) The expression $9x^3-15x^2=3x^2(3x-5)$ is an identity.

b) If we multiply two 1st-degree and 2nd-degree binomials, we get a 3rd-degree polynomial.

c) If we add two binomials, we always get a binomial.

d) Numbers are monomials.

e) Monomials $3a^2b$ y $-3ab^2$ are similar.

f) When we divide $3x^2y^2:6xy^2$, we get a monomial.

101

Get informed

- These two texts complement the introductory text from the beginning of the unit.

Investigate

- This activity aims to present learning as a process of discovery: observe, touch, verify, analyse, discover patterns, hypothesise, generalise and so on.
- First, make sure that students understand the structure well, analysing it as a whole class. Students should be able to discover the law of formation of the first triangle and be able to extend the triangle by a few rows. Then, either individually or in groups, students should answer the remaining questions.

  In the second triangle, the sum of the elements in each row is equal to the power 2. Ask students to check this in a few of the rows.

  The third stage is the most challenging one. The key here is to realise that the numbers are equal to the successive sums of the first natural numbers: $a_n = 1 + 2 + 3 + \ldots + n$. Once they know this, students apply the procedures they have learned to add the terms in an arithmetic progression.

Answers

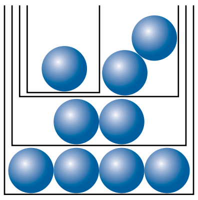
-

S_1	S_2	S_3	S_4	S_5	…	S_n
2	4	8	16	32	…	S^n

- | 1 | 6 | 21 | | 1 | 20 | 210 |
- | 1 | n | $\frac{(n+1)\cdot n}{2}$ |

Train yourself by solving problems

Solutions

- The first cyclist will be 10 km ahead of the second cyclist.
- 
- 8 (out of 30) people don't like rock or electronic music.

Answers to 'Self-assessment'

1 a) $\frac{15+7x}{12}$

b) If x is the price of a coffee, $\frac{11}{2}x$

c) Area = $2x^2 + 20x$; Volume = $5x^2$

2 2 a) $7x^3 - 5x^2 + x$ b) $x^2 - 16x$

3 3 $-18x^2 + 59x - 26$

4 $\frac{2x-3}{2x+3}$

5 a) Quotient: $3x^2 - 4x + 6$; Remainder: $-6x + 4$

b) Quotient: $x^2 + x - 4$; Remainder: 10

6 a) $\frac{-x^2+5x+6}{2x^2}$ b) $\frac{3}{x+1}$

7 $m = -7$

8 a) T b) T c) F d) F e) F f) T

Answers to 'Think and practise' (page 89)

7 a) -8 b) $-12x$

c) $-23x$ d) $10x^2 + 7x - 17$

e) $2x^2 - 25$ f) $12x - x^2$

8 $12x^3 - 8x^5 + 4x^2y^2 - \frac{4}{3}x^2 = 4x^2\left(3x - 2x^3 + y^2 - \frac{1}{3}\right)$

$(x^2 - 1) + (x^2 - 2x + 1) - (4x - 4) = (x - 1)(2x - 4)$

$6(x^2 - 4x + 4) - (2x^2 - 8) + (30x - 60) = 2(x - 2)(2x + 7)$

$9x^2 - 18xy^2 - 6xyz + 6x = 3x(3x - 6y^2 - 2yz + 2)$

9 a) $x - 2$ b) $2x - 10$

c) $-20x - 24$ d) $-13x + 63$

e) $-13x + 821$ f) $9x^2 + 76x + 155$

Answers to 'Exercises and problems' (page 97)

31 a) $\frac{x}{3}$ b) $\frac{x}{3}$ c) 4

d) $\frac{1}{x+5}$ e) $2x$ f) $\frac{x-2}{3}$

32 a) $\frac{2}{3x}$ b) $\frac{1}{x-1}$ c) $\frac{1}{x-2}$

d) $\frac{x}{x+3}$ e) $\frac{x-2}{x+2}$ f) $\frac{x(x+1)}{3}$

33 a) $\frac{x+2}{x^2}$ b) $\frac{11}{6x}$ c) $\frac{5x-6}{2x^2}$

d) $\frac{-x^2+4x-1}{x^2}$ e) $\frac{2x^2-2x+3}{x-1}$ f) $\frac{x-x^2}{x+1}$

34 a) $\frac{x^2+2x-3}{6x^3}$ b) $\frac{x^2+x-14}{x^2-7x}$

c) $\frac{x^2-8}{x^2-4x}$ d) $\frac{x^2+10x-3}{x^2-9}$

e) $\frac{x^2+x+10}{4(x-1)}$ f) $\frac{2x^2+5x+2}{x(x+1)}$

35 a) $\frac{1}{3}$ b) $\frac{x}{2}$ c) $\frac{3}{2(x-2)}$

d) $\frac{x-1}{x}$ e) $\frac{5}{x-1}$ f) $\frac{x}{5}$

36 a) $\frac{3}{(x+1)x}$ b) $\frac{3x^2+2x}{x^2-1}$

c) $\frac{3}{2(x-1)}$ d) $\frac{2}{x-1}$

37 a) $2x - 4$ b) $x^2 + 3x$ c) $\frac{x-6}{x}$

d) $\frac{2}{x}$ e) $\frac{2}{(x-2)}$ f) $\frac{1}{(x+3)}$

Notes